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Graphical Solutions to Optical Problems of Three Different Types

ERIC J. IRONS
Queen Mary College, London, E.1., England
(Received November 29, 1951)

The following problems are considered:

I. Paraxial ray tracing by the focal plane method.—For spherical refracting and reflecting surfaces it is shown that by drawing two secondary axes, one parallel to a ray incident at an arbitrary angle and the other parallel to the path of the same ray after refraction or reflection, the positions of the focal planes are readily established and their necessary properties deduced. The same is done for a lens by a method based on the fact that a focal plane of the first surface is conjugate to a focal plane of the second surface.

II. The variation in the distance L between the conjugate foci of a thin lens (focal length f) with the distance l of one focus from the lens.—A general formula is established, and a simple construction for plotting the graphs between L/f and l/f is described.

III. Refraction by prism combinations and the prism refractometer.—A graphical method is used to discuss the design of the Amici prism, the direct vision spectroscope, and achromatic prisms giving appreciable deviations. A graphical method is also described for finding the refractive index of a liquid from results obtained with a prism refractometer.

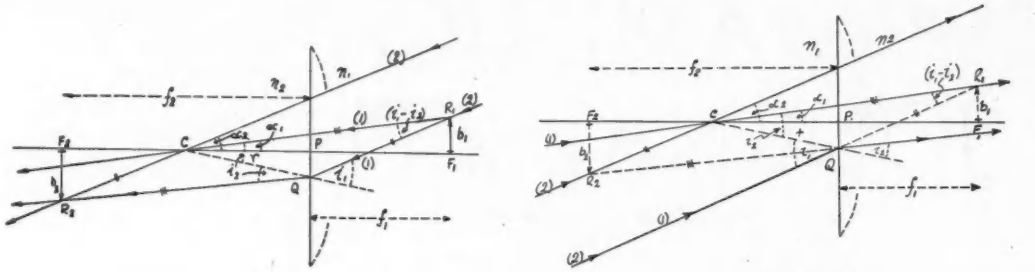
I. PARAXIAL RAY TRACING BY THE FOCAL PLANE METHOD

THE method is commonplace; its justification is not. Accordingly the positions and properties of focal planes are investigated for spherical refracting and reflecting surfaces, and the results obtained for the former are applied to justify the method for lenses. The diagrams are arranged to indicate the correspondence between the effect of a converging system on a diverging beam and a diverging system on a converging beam; and, as usual, distances perpendicular to the principal axes are magnified.

In each diagram of Figs. 1 and 2 two rays [marked (2)] of a paraxial pencil are shown coming from the tip of an object which is so distant that the rays may be regarded as parallel. One of these rays is directed towards the center of curvature C of the surface (and is, in consequence, refracted without deviation or reflected back along its own path), while the deviation of the other depends on the angle it makes with the normal, i.e., the radius, at the point of incidence. These two rays finally converge to or diverge from R_2. Two paraxial rays [marked (1)], originally diverging from or converging towards an object R_1, and subsequently rendered parallel by refraction or reflection, are also shown; one of these also passes through C. For convenience the ray not associated with C is common to both pencils.

For the *refracting surfaces* (Fig. 1), if R_1Q and CR_2 show the direction[1] of two incident parallel rays from a distant object, and QR_2 and R_1C the direction[1] of two rays rendered parallel

[1] But *not* the sense for the surface of negative power.



(a)

(a) Converging surface [Concave to medium of greater refractive index ($n_2>n_1$)]. Power is positive because a parallel beam converges after refraction to form a real image in the second focal plane. Power $=+n_2/f_2$ (by definition) $=n_1/f_1$ [by Eq. (1)] $=(n_2-n_1)/(f_2-f_1)$ (by algebra) $=(n_2-n_1)/r$ [by Eq. (2)]. If the paths of the rays are reversed, R_1F_1 and R_2F_2 become, respectively, the second and first focal planes, the power ($=+n_1/PF_1$) being unchanged.

(b)

(b) Diverging surface. (Convex to medium of greater refractive index. ($n_2>n_1$)). Power is negative because a parallel beam diverges after refraction to form a virtual image in the second focal plane. Power $=-n_2/f_2$ (by definition) $=-n_1/f_1$ [by Eq. (1)] $=-(n_2-n_1)/(f_2-f_1)$ (by algebra) $=-(n_2-n_1)/r$ [by Eq. (2)]. If the paths of the rays are reversed, R_1F_1 and R_2F_2 become, respectively, the second and first focal planes, the power ($=-n_1/PF_1$) being unchanged.

FIG. 1. Focal planes of refracting surfaces.

by refraction, then R_2CR_1Q is a parallelogram with $CR_1=QR_2$ and $CR_2=QR_1$. It follows that, in ΔQR_1C,

$$QR_1/CR_1(=QR_1/QR_2)=\sin i_2/\sin i_1=n_1/n_2.$$

Moreover, as $\cos\alpha_1=PF_2/QR_2$ and $\cos\alpha_2=PF_1/QR_1$,

$$PF_1/PF_2=(QR_1/QR_2)(\cos\alpha_2/\cos\alpha_1)$$
$$=(n_1/n_2)(\cos\alpha_2/\cos\alpha_1),$$

where R_1F_1 and R_2F_2 are perpendicular to the axis F_1PF_2. But as $\alpha_1\to 0$, $\cos\alpha_1\to 1$, and $R_1\to F_1$; and as $\alpha_2\to 0$, $\cos\alpha_2\to 1$ and $R_2\to F_2$, so that F_1 and F_2 must be the focal points and

$$f_1/f_2=n_1/n_2. \tag{1}$$

Moreover, as CR_1 and QR_2 are equal and inclined at α_1 to the axis, $PF_2=CF_1$, or

$$f_2-f_1=r. \tag{2}$$

Thus, the distances of the focal planes from the surfaces are calculable from the values of n_1, n_2, and r.

Again, for paraxial rays,

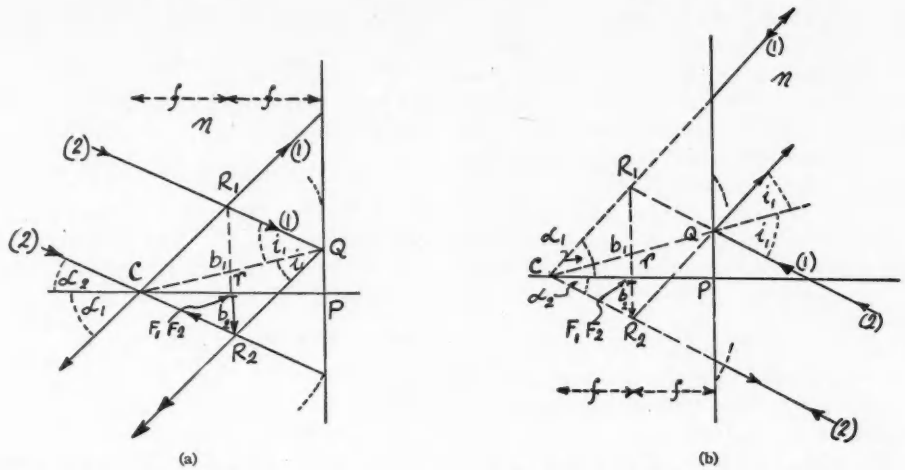
$$b_1=CF_1\tan\alpha_1=f_2\tan\alpha_1 \text{ and}$$
$$b_2=CF_2\tan\alpha_2=f_1\tan\alpha_2. \tag{3}$$

Thus, the distances of R_1 and R_2 from the axis are independent of i_1 and depend only on α_1 and α_2, respectively. It follows that all rays (of paraxial pencils) parallel to those shown are associated with R_1 or with R_2.

For the *reflecting surfaces* (Fig. 2), the same arguments, simplified by the fact that the angles of incidence and of reflection are equal, may be applied; they lead to similar conclusions including the results $f_1=f$, say, $f_2+f_1=r$, $b_1=f\tan\alpha_1$ and $b_2=f\tan\alpha_2$ corresponding to Eqs. (1), (2), and (3), respectively.

Thus, to find the subsequent path of any paraxial ray incident at a point Q on a spherical refracting or reflecting surface, *either* (A) mark the point of intersection R_1 of the incident ray (produced if necessary) with the first focal plane, join R_1 to C, the center of curvature of the surface, and then draw a line from Q parallel to the secondary axis R_1C (or CR_1); *or* (B) draw the secondary axis parallel to the incident ray and mark its point of intersection R_2 with the second focal plane; then QR_2 (or R_2Q for a surface of negative power) is the required direction.

The justification for the process of ray tracing through a *lens* is based on procedures A and B for a refracting surface. Inasmuch as a lens has, in general, finite thickness and terminates in two coaxial spherical refracting surfaces, the procedures may be applied either (a) directly to each lens surface in turn, or (b) to locate the focal planes *of the lens* and derive a new procedure based upon the properties of these planes.



(a) Converging (concave) surface. Power is positive as a parallel beam converges after reflection to form a real image in the second focal plane. Power $= +n/f = +2n/r$.

(b) Diverging (convex) surface. Power is negative as a parallel beam diverges after reflection to form a virtual image in the second focal plane. Power $= -n/f = -2n/r$.

FIG. 2. Focal planes of reflecting surfaces.

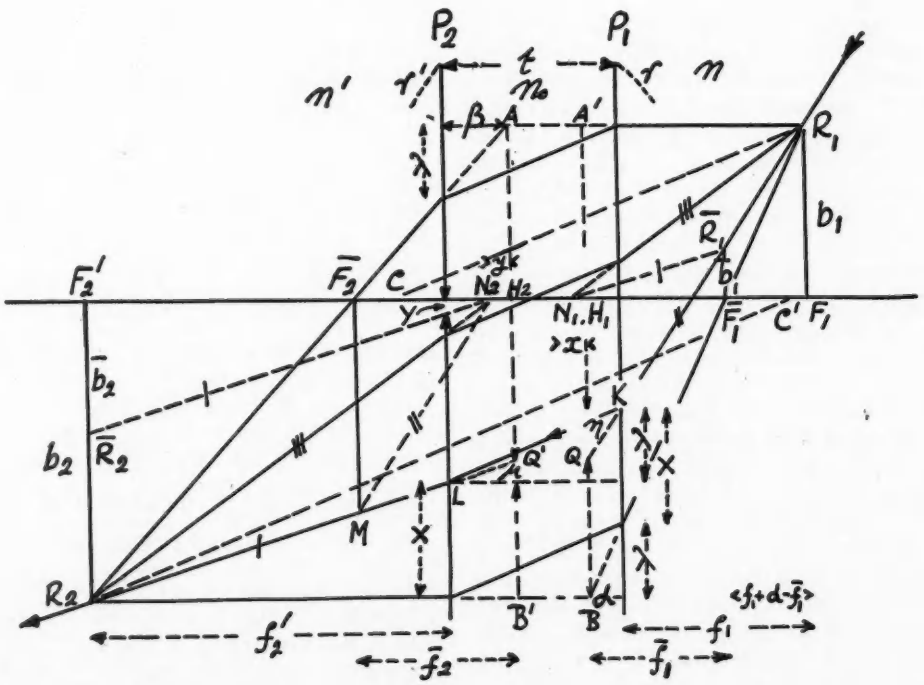


FIG. 3. Ray tracing through lens of positive power having refracting surfaces of positive powers P_1 and P_2.

Method (a) is more direct when values of the refractive indices and radii of curvature are known, but method (b) has to be applied when only the focal and principal points of the lens are specified. The discussion of method (b) is based on method (a) (which does not, therefore, require a separate treatment) and includes the derivation of the usual "thick lens" formulas. This discussion is limited to a lens whose power and that of its refracting surfaces are all positive. The limitation seems justified as the results obtained for this one system are shown to apply to the simpler systems normally encountered, and the "thick lens" formulas are better derived by an analytical method without any assumption concerning the signs of the powers.

In Fig. 3, the surfaces (radii of curvature r, r'; centers of curvature C, C') separate media having refractive indices n, n_0, and n' such that $n_0>n'>n$. The positions of the first focal point F_1 of the first surface, and the second focal point F_2' of the second surface are known by inserting the values of n, n_0, n', r, and r' in the formulas appended to Fig. 1; and it is required to trace through the system a ray R_1K which passes through R_1 at a height b_1 above F_1. Accordingly KL is drawn parallel to R_1C and L is joined to R_2, where the line $C'R_2$, which is parallel to KL, cuts the focal plane $F_2'R_2$. Remembering that all rays from R_1 are rendered parallel to each other by the first refraction and therefore directed towards R_2 by the second refraction, the first and second focal planes *of the lens* are located at $\bar{F}_1$ and $\bar{F}_2$, respectively, by rays which also determine the positions of the principal planes containing H_1 and H_2. Finally, from R_1 is drawn a ray R_1N_1 such that, on emerging from the lens, it proceeds in a parallel direction N_2R_2.

It is required to derive expressions for α, β, $\bar{f}_1$, $\bar{f}_2$, H_1N_1, and H_2N_2 (see Fig. 3) and establish the proposition that LR_2 passes through M (the point on the second focal plane of the lens such that N_2M is parallel to R_1K) and is parallel to $\bar{R}_1N_1$, where $\bar{R}_1$ is the point of intersection of the ray with the first focal plane of the lens.

As the triangle having sides λ and α (subsequently called $\triangle\lambda\alpha$) is similar to $\triangle A'BR_1$, and using a similar notation, $\triangle\lambda\beta$ is similar to $\triangle AB'R_2$,

$$(b_1+b_2)/\lambda=(f_1+\alpha)/\alpha=(f_2'+\beta)/\beta,$$

so that, having regard to the legend of Fig. 1,

$$\alpha/\beta=f_1/f_2'=(n/P_1)(P_2/n'). \tag{4}$$

By construction R_1C, $C'R_2$, and KL are parallel, and therefore,

$$b_1/(r+f_1)=b_2/(r'+f_2')=\lambda/t.$$

These results, taken in conjunction with that contained in Eq. (2), and additional relations derived from the similar triangles $\lambda\alpha$ and $\bar{F}_1H_1B$, and $\lambda\beta$ and $\bar{F}_2H_2A$ show that

$$\left.\begin{aligned}\alpha=\bar{f}_1(\lambda/b_2)&=\bar{f}_1t/(r'+f_2')\\&=\bar{f}_1t/f_1'=\bar{f}_1P_2t/n_0\\\text{and}\\\beta=\bar{f}_2(\lambda/b_1)&=\bar{f}_2t/(r+f_1)\\&=\bar{f}_2t/f_2=\bar{f}_2P_1t/n_0,\end{aligned}\right\} \tag{5}$$

where f_1' and f_2 are the (unmarked on Fig. 3) first focal length of the second surface and the second focal length of the first surface. Hence, $\alpha/\beta=(P_2/P_1)(\bar{f}_1/\bar{f}_2)$, and, by comparison with Eq. (4), $\bar{f}_1/\bar{f}_2=n/n'$. Thus,

$$\left.\begin{aligned}&n/\bar{f}_1=n'/\bar{f}_2=P, \text{ say,}\\&\text{(the power of the lens, since } \bar{f}_1 \text{ and } \bar{f}_2\\&\quad\text{are, by convention, the focal lengths)}\\&\alpha/n=(t/n_0)(P_2/P), \text{ and}\\&\beta/n'=(t/n_0)(P_1/P).\end{aligned}\right\} \tag{6}$$

Again, putting $H_1N_1=x$ and $H_2N_2=y$ and considering the similar triangles $N_1R_1F_1$ and $N_2R_2F_2'$, $\bar{F}_1H_1B$ and $\bar{F}_1F_1R_1$, and $\bar{F}_2H_2A$ and $\bar{F}_2F_2'R_2$,

$$\frac{b_1}{b_2}=\frac{f_1+\alpha+x}{f_2'+\beta-y}=\frac{f_1+\alpha-\bar{f}_1}{\bar{f}_1}=\frac{\bar{f}_2}{f_2'+\beta-\bar{f}_2}, \tag{7}$$

whence

$$\frac{b_1}{b_2}=\frac{f_1+\alpha+(\bar{f}_2-\bar{f}_1)}{f_2'+\beta-(\bar{f}_2-\bar{f}_1)}. \tag{8}$$

Thus, Eqs. (7) and (8) are satisfied by

$$x=y=(\bar{f}_2-\bar{f}_1)=\text{constant}. \tag{9}$$

This indicates that N_1 and N_2 are fixed points independent of the direction which the incident light makes with the axis; thus, e.g., a ray R_1N_1 is refracted in a parallel direction N_2R_2 through N_2, so that N_1 and N_2, like the centers of curvature of refracting and reflecting surfaces, are nodal points.

Then, substituting $f_1=n/P_1$, $f_2'=n'/P_2$, and the values of α, β, $\bar{f}_1$, and $\bar{f}_2$ contained in Eq. (6) in the last two values of b_1/b_2 given in Eq. (7), readily leads to

$$P=P_1+P_2-(t/n_0)P_1P_2. \tag{10}$$

(Equations (10), (6), and (9) fix the positions of the cardinal points of the lens in terms of the refractive indices, t, and P_1 and P_2 (or, more basically, r and r'); this permits ray tracing by method (b) when the cardinal points have not been located experimentally.)

Again, by similar triangles,

$$\frac{X+Y}{\bar{b}_2}=\frac{X}{b_1}\frac{\bar{b}_1}{\bar{b}_2}+\frac{Y}{\bar{b}_2}$$

$$=\frac{f_1}{(f_1+\alpha-\bar{f}_1)}\cdot\frac{(\bar{f}_1+x)}{(f_2'+\beta-x)}+\frac{\beta-x}{f_2'+\beta-x}=1,$$

since, by Eqs. (4), (5), (7), and (9), $\alpha/\beta=f_1/f_2'=(\bar{f}_1/\bar{f}_2)(b_1/b_2)=(f_1+\alpha-\bar{f}_1)/\bar{f}_2=(f_1+\alpha-\bar{f}_1)/(\bar{f}_1+x)$, and, therefore, $f_1(\bar{f}_1+x)/(f_1+\alpha-\bar{f}_1)=f_2'$. It follows that $(b_2-\bar{b}_2)=b_2-(X+Y)$, and LR_2 is parallel to $N_2\bar{R}_2$ (and also $N_1\bar{R}_1$).

The unit magnification property of the principal planes is established by using ray R_1KLR_2 and equating values for the gradients of R_1KQ and of $Q'LR_2$ (or $\bar{R}_1N_1$), whence

$$\eta+\mu=\frac{\alpha(b_1-\bar{b}_1)}{f_1+\alpha-\bar{f}_1}+\frac{\beta\bar{b}_1}{\bar{f}_1+x}$$

$$=\frac{\bar{f}_1\lambda(b_1-\bar{b}_1)b_2}{b_2\bar{f}_1b_1}+\frac{\bar{f}_2\lambda\bar{b}_1}{b_1\bar{f}_2}=\lambda,$$

by Eqs. (5), (7), and (9). Thus, if the left-hand side of the drawing were moved towards the right so that H_2 coincided with H_1, Q' would coincide with Q and $\bar{R}_1N_1$ and $N_2\bar{R}_2$ would become collinear. Figure $\bar{R}_1N_1MQ'$ would become a parallelogram, since $\bar{R}_1N_1$ is parallel to $Q'M$, and by Eq. (9), the projections of these lines on the axis are both $\bar{f}_2$. It follows that N_2M is parallel to $\bar{R}_1K$, the proposition is proved, and the path of the ray may be traced by utilizing the properties of the first and second focal planes of the lens and the nodal points.

If the *lens is thin* (i.e., $t=\alpha=\beta=0$, H_1 and H_2 coincide in "the plane of the lens" at H, and N_1 coincides with N_2 at N, Fig. 4), the path of the ray after refraction is known *either* by drawing a line parallel to $\bar{R}_1N$ (or $N\bar{R}_1$) *or* by joining KM (or producing MK), where NM is parallel to the incident ray. Distance NH is still $(\bar{f}_2-\bar{f}_1)$, as may be observed by substituting $\alpha=\beta=0$ in Eqs. (7) and (8), but it should be observed that NH is an essentially positive quantity and equal to $(\bar{f}_1-\bar{f}_2)$ if $\bar{f}_1>\bar{f}_2$. Thus, N (Fig. 4) and N_1 and N_2 (Fig. 3) are situated on the same side of H, H_1, and H_2, respectively, as the longer focal length, and not necessarily on the same side as the material of greater refractive index n'.

If, in addition to the lens being thin, $n=n'$, then $\bar{f}_2=\bar{f}_1$ by Eq. (6), and $NH=0$, i.e., N co-

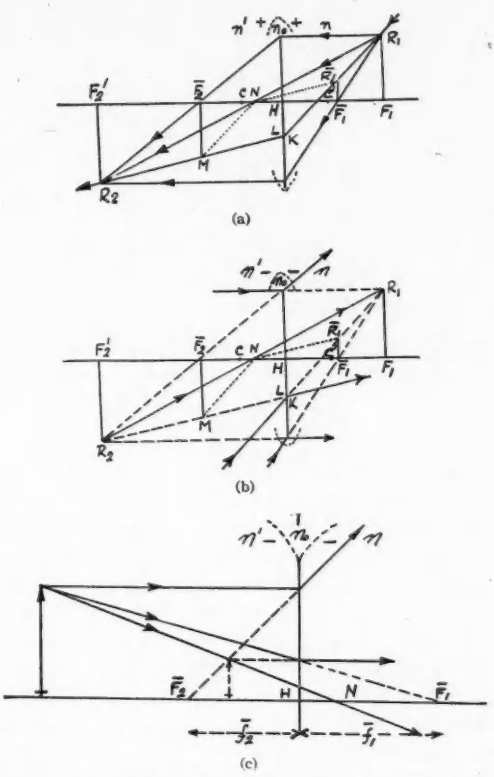
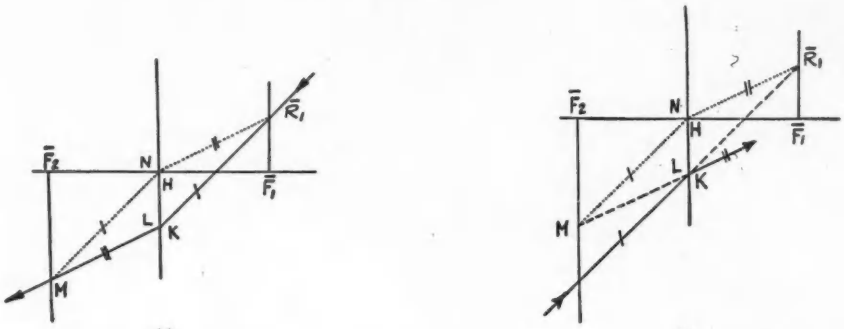


FIG. 4. Ray tracing through thin lenses. $n\neq n'$. (a) Lens with both refracting surfaces of positive power. $n_0>n'>n$. (b) Lens with both refracting surfaces of negative power. $n>n'>n_0$. In the above diagrams the rays associated with R_1 are rendered parallel to R_1C and R_2C' by the first refraction and converge to or diverge from R_2 when refracted at the second surface, i.e., R_1 and R_2 are conjugate foci. (c) $n_0>n'>n$, $\bar{f}_1>\bar{f}_2$. Example of nodal point on opposite side of H to material of greater refractive index, but, as always, on same side of H as longer focal length.



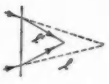
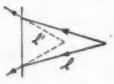
(a) Lens of positive power. (b) Lens of negative power.

FIG. 5. Ray tracing through thin lenses. $n=n'$. Similarly marked lines are parallel.

incides with H and the drawing of an undeviated ray through the center of the lens is justified. Ray tracing through a lens may then be effected by one of the relatively simple processes illustrated in Fig. 5. It is sometimes sought to justify these constructions by stating that rays directed to or from a point $\bar{R}_1$ on a focal plane are parallel either before or after refraction. Strictly, this should be preceded by two demonstrations: one showing that the property of the principal focus for paraxial rays is shared by all points on the focal "plane," and the other that N is situated at the lens "center." In this last connection it may be remarked that the "optical center of a lens"[2] O seems to have a somewhat limited application in ray tracing, since either the lens is "thin" and O coincides with N_1 and N_2, or it is "thick" and more direct use can be made of the separated nodal points.

To trace a ray through two separated thin lenses, whether surrounded by the same material or not, it is usually simpler to use one focal plane of each lens, rather than to calculate the positions of the cardinal points of the system by substituting the powers of the lenses for P_1 and P_2 in Eqs. (6) and (10).

TABLE I. Refraction of converging and diverging beams by a thin lens.

	Converging (+ve power) lens			Diverging (−ve power) lens		
						
	$+\frac{1}{l'}=-\frac{1}{l}+\frac{1}{f}$	$-\frac{1}{l'}=-\frac{1}{l}+\frac{1}{f}$	$+\frac{1}{l'}=+\frac{1}{l}+\frac{1}{f}$	$-\frac{1}{l'}=+\frac{1}{l}-\frac{1}{f}$	$+\frac{1}{l'}=+\frac{1}{l}-\frac{1}{f}$	$-\frac{1}{l'}=-\frac{1}{l}-\frac{1}{f}$
	$L=l+l'$	$L=l'-l$	$L=l-l'$	$L=l+l'$	$L=l'-l$	$L=l-l'$
	$\frac{L}{l(L-l)}=\frac{1}{f}$	$\frac{L}{l(L+l)}=\frac{1}{f}$	$\frac{L}{l(-L+l)}=\frac{1}{f}$	$\frac{L}{l(-L+l)}=\frac{-1}{f}$	$\frac{-L}{l(L+l)}=\frac{-1}{f}$	$\frac{L}{l(L-l)}=\frac{-1}{f}$
a	−	−	+	+	+	−
	$(l/f>1)$	$(l/f<1)$		$(l/f>1)$	$(l/f<1)$	
b	+	−	+	−	+	−
L/f	y_1-y_2	$-y_1+y_2$	y_1-y_2	y_1-y_2	$-y_1+y_2$	y_1-y_2

[2] The point through which pass rays that are undeviated although laterally displaced, and which divides the line joining the centers of curvature of the lens, faces either internally or externally in the ratio of the radii of curvature.

II. THE VARIATION IN THE DISTANCE BETWEEN THE CONJUGATE FOCI OF A THIN LENS WITH THE POSITION OF ONE OF THEM

Table I is concerned with the refraction of converging and diverging beams incident on converging and diverging lenses of focal length f. Putting L for the distance between the initial and final foci, the relations shown under the several diagrams are summarized in the equation

$$\frac{bL}{al(-bL\pm l)}=\pm\frac{1}{f},$$

in which the positive or negative sign is taken according to the power of the lens, $a=\pm 1$ according to whether the light initially converges or diverges, and $b=\pm 1$ according to whether the light finally converges or diverges. This equation may be rewritten

$$\pm\frac{bL}{f}=\frac{l}{f}-\frac{1}{\{1/(l/f)\}\pm a},$$

which is a form convenient for investigating graphically the relation between the dimensionless and essentially positive ratios L/f and l/f. Thus, on Fig. 6 are plotted $y_1=l/f$, $y_2=[\{1/(l/f)\}+1]^{-1}$, and $y_3=[\{1/(l/f)\}-1]^{-1}$, the corresponding coordinates of the last two curves being found from the simultaneous values of the intercepts on the axes of a series of straight lines

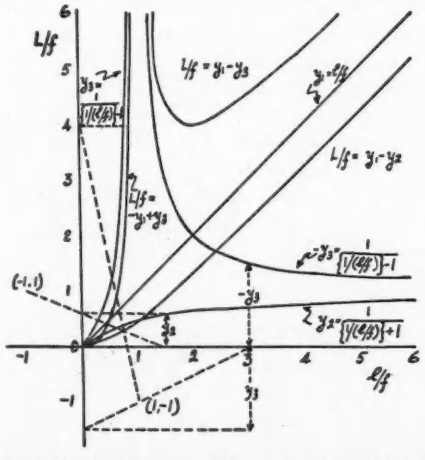


FIG. 6. Graph of distance between conjugate foci/focal length against distance of one focus from lens/focal length for any thin lens.

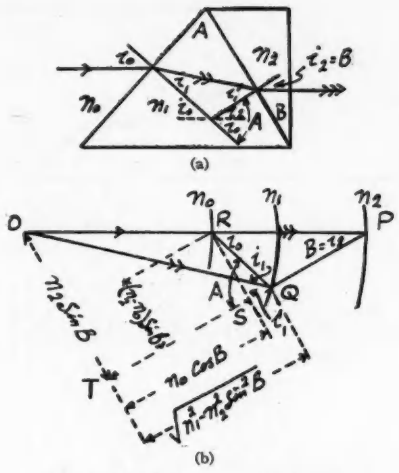


FIG. 7. The design of an Amici prism.

passing through the points $(-1, 1)$ and $(1, -1)$, respectively. [This is justified since, by similar triangles, $y_2/1=(l/f)/\{(l/f)+1\}$ and $y_3/1=(l/f)/\{(l/f)-1\}$ for $l/f>1$ or $y_3/(y_3+1)=(l/f)/1$ for $l/f<1$.] The curves representing y_1, y_2, and y_3 have been used to construct others corresponding to $L/f=y_1-y_2$, $L/f=y_1-y_3$ (for $l/f>1$) and $L/f=-y_1+y_3$ (for $l/f<1$). These last curves demonstrate the relations sought, and it may be observed that a minimum value of L $(=4f=2l=2l')$ occurs with both types of lens.

III. REFRACTION BY PRISM COMBINATIONS AND THE PRISM REFRACTOMETER

The treatment in this section is based on a geometrical construction previously used to discuss refraction by a single prism.[3] This construction involves the drawing of arcs whose radii represent refractive indices, but, to clarify the diagrams, the ratios of these radii are not typical of the ratios of the indices. The necessity for this is very apparent with the diagrams of the direct vision spectroscope and achromatic prism, where, because of the small change in refractive index with color, arcs would otherwise be crowded together. Moreover, on account of this small variation, information derived from large scale drawings of Figs. 8 and 9 is unlikely to be very accurate, although it may well economize time in the initial stages of a com-

[3] E. J. Irons, Phil. Mag. **34**, 608 (1943).

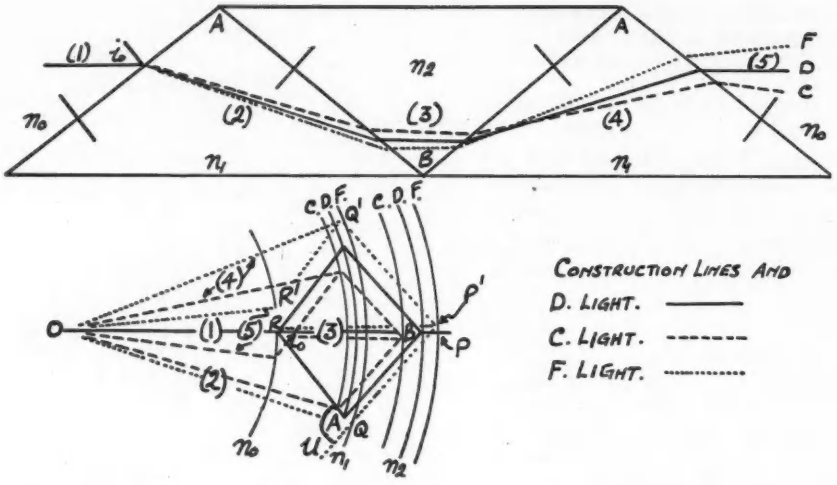


FIG. 8. Ray tracing through a direct vision spectroscope.

putation which proceeds by successive approximations.

(A) *The design of the Amici Prism and ray tracing through a direct vision spectroscope.*[4]—In the Amici prism arrangement (Fig. 7a) the angles A and B are such that light, having refractive indices n_0, n_1, and n_2 in the media concerned, emerges parallel to its incident direction and perpendicular to the face of the second prism. To determine A in terms of B (supposed known), a line ORP, representing the incident ray (Fig. 7b) is drawn from O to cut the arcs of center O and radii n_0 and n_2 in R and P, and $\angle B = \angle OPQ$ is set off on OP so that PQ (which represents the direction of the normal at the interface between the two prisms) cuts the arc of center O and radius n_1 in Q. Then $\angle RQS = \angle A$, where RS, like OT, is perpendicular to PQ. The proof is as follows: In $\triangle OPQ$, $OP/OQ = n_2/n_1 = \sin OQS/\sin i_2$ by construction and by geometry, respectively, so that $\angle OQS = \angle i_1'$ by the second law of refraction. From Fig. 7a, i_0 and i_1 must satisfy the equations $A = i_1 + i_1' = i_0 + i_2 = i_0 + B$ and $n_0 \sin i_0 = n_1 \sin i_1$, and, by reference to Fig. 7b, it may be observed that putting $\angle QRP = \angle i_0$ and $\angle RQO = \angle i_1$ fulfills these conditions. This shows that $\angle A = \angle RQS$; RQ represents the normal at the first surface; and, from the dimensions marked on Fig. 7b,

$$\tan A = \frac{(n_2 - n_0)\sin B}{\sqrt{(n_1^2 - n_2^2 \sin^2 B)} - n_0 \cos B}.$$

(Note that if Q lies between S and T, the denominator of this expression is negative, and $\angle A$ is obtuse as in Fig. 8.)

Smith[5] derived the above formula analytically and observed that, as light of any color other than that considered will suffer a deviation which is a function of the corresponding refractive indices, the Amici prism constitutes the most

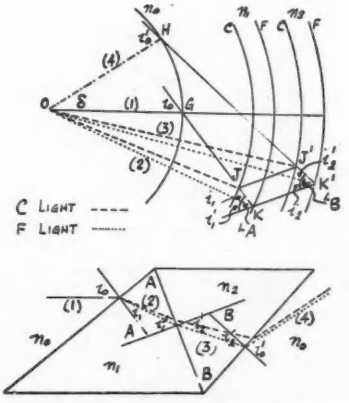


FIG. 9. Design of an achromatic prism producing a large deviation.

[4] J. H. Dowell, Proc. Optical Convention **II**, 965–981 (1926) has considered direct vision prisms by essentially the same method as that employed here, except that he deals with wave fronts instead of rays.

[5] C. J. Smith, Phil. Mag. **33**, 775 (1942), especially p. 804.

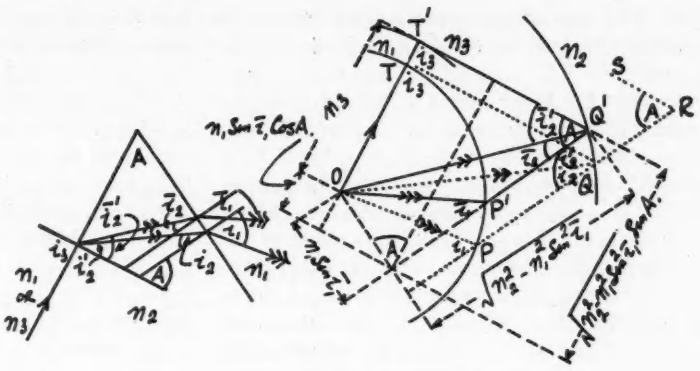


FIG. 10. Graphical determination of refractive index by the prism refractometer. $A = \bar{\imath}_2 + \bar{\imath}_2'$.

simple form of direct vision spectroscope. He also noted that an arrangement of two identical Amici prisms (i.e., three prisms in all, Fig. 8) increases the dispersion and eliminates the lateral displacement.

Ray tracing through such a compound prism is readily performed as shown in the lower diagram of Fig. 8. Here are indicated (a) the construction lines by which the value of the $\angle A$ has been derived from a knowledge of $\angle B/2$ and n_0, n_1 and n_2 for sodium light; (b) additional full lines to represent normals to the third and fourth refracting surfaces (i.e., parallels to $P'Q'$ and $Q'R'$, respectively); and (c) arcs representing refractive indices for, say, C and F light. The directions of rays corresponding to yellow light are omitted, but the successive directions of the rays for C and F light are indicated and numbered to correspond with the upper diagram of Fig. 8. Thus, for the F light, direction (1) and $\angle PRQ = \angle i_0$ having already been fixed, OQ represents direction (2) since, in the nomenclature of Fig 7a, $OQ/\sin i_0 = OR/\sin OQR = n_0/\sin i_1$. Again, QP' being perpendicular to the second refracting surface and $\angle OQU = \angle i_1'$, $OP'/\sin i_1' = OQ/\sin OP'Q$, so OP' gives direction (3); directions (4) and (5) are determined similarly.

In practice, an odd number of prisms in excess of three may be used, but, although the construction is readily extended, the diagram becomes complicated.

(B) *The achromatic prism* is a compound prism designed to produce deviation without, as far as possible, any dispersion (Fig. 9).—When the required deviation (δ) is small, the refracting angles A and B of the component prisms are readily estimated by assuming that the deviation is independent of the angle of incidence. Larger values of δ require larger values of A and B and a specified angle of incidence; some indication of the values of A and B may then be derived from a large scale drawing of the type shown in the upper diagram of Fig. 9. Here the concentric arcs have radii representing the refractive index of air and the refractive indices of the two glasses for the two colors (say those of the C and F lines) for which the prism is to be achromatized.

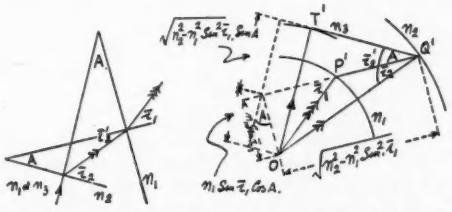


FIG. 11. Graphical determination of refractive index by the prism refractometer. $A = \bar{\imath}_2 - \bar{\imath}_2'$.

Line OG represents the direction of the incident light and $\delta (= \angle GOH)$ is set off equal to the required common angle of deviation. Line GJK represents the normal at the point of incidence, and OJ and OK the directions of the refracted rays in the first prism. It is then required to draw through J and K two parallel lines JJ' and KK' which cut the arcs representing the refractive indices of the glass of the second prism at J' and K' and make H, J', and K' collinear; this is readily effected by drawing a number of trial lines through H and using a parallel-rule to test the parallelism of JJ' and KK'. Alter-

natively, if one of the prism angles (say A) is specified, $\angle i_0$ may be varied, $\angle A$ marked off at K, and JJ', KK', and $HJ'K'$ located by trial. This done, the lower figure is derived from the upper. The construction is justified in the same way as before, and may be checked for F light from the similarly marked angles in the figures.

(C) *The prism refractometer.*—Figure 10 will recall the variation of Wollaston's method for determining the refractive index of a liquid by using a prism of refracting angle A. Light enters one refracting face of the prism (or enters and is scattered from the face opposite the refracting edge), and the direction which separates the light and dark fields of view is located at $\angle(90 \pm i_1)$ to the second refracting face. If the values of i_1 (say, i_1 and $\bar{\imath}_1$, respectively) are found when those of i_2' correspond to the critical angles first for glass and air (i_2'), and second for glass and liquid ($\bar{\imath}_2'$), the refractive indices of the glass and liquid may be found by the method shown in the right-hand diagram of Fig. 10. Here OP (P being the point of intersection with the arc of center O whose radius represents n_1) is the direction of the emergent ray at $\angle i_1$ to PQ, a line parallel to the normal. The $\angle PRS = \angle A$ is set off on PQ, and Q is located by drawing parallel to RS the tangent QT to the arc of radius n_1. Then, since the three equations $OQ/\sin i_1 = n_1/\sin i_2$ (from $\triangle OPQ$), $OQ/\sin 90 = n_1/\sin i_2'$ (from $\triangle OQT$) and $A = i_2 + i_2'$ are satisfied, OQ represents n_2 in length and gives the direction of the "critical ray" in the glass, $\angle OQP = \angle i_2$, and $\angle OQT = \angle i_2'$. An arc of radius OQ is now constructed, and OP' is drawn to make $\bar{\imath}_1$ with $P'Q'$, which, representing a second normal to the surface, is parallel to PQ and cuts the arc of radius OQ at Q'. From Q', $Q'T'$ is drawn parallel to QT to cut OT produced in T', and, by similar reasoning to that given above, OT' represents n_3 on the same scale as OT represents n_1. Figure 11 is similar to Fig. 10, but the emergent ray is on the opposite side of the normal and the construction for determining n_2 is omitted. From the magnitudes marked on the diagrams there results the usual formula

$$n_3 = \sin A \sqrt{(n_2^2 - n_1^2 \sin^2 \bar{\imath}_1)} - \cos A (n_1 \sin \bar{\imath}_1),$$

in which the value of $\bar{\imath}_1$, and therefore $\sin \bar{\imath}_1$, must be prefaced by a $\pm$ sign according as the conditions are those of Fig. 10 or Fig. 11.

The whole system of Nature is one immense series of causes and effects, whose beginning and end are equally hid in the depths of infinity. Only a small, a very small, portion of it comes under our immediate observation; being exposed alike to the sight and other senses of all mankind. Almost every phaenomenon *is, at once, the cause of manifold effects; and one effect, among many, of a superior cause. The business of Science is to extend our views, by unfolding the latent causes which exist in Nature; and thence explaining their manifest effects. The discovery of one such real cause, unknown before, if it be of general or very extensive influence, as that of universal gravity, is to be esteemed a great advancement of natural philosophy. To undervalue such a discovery, as some have done, because the cause of that cause cannot yet be assigned, is highly absurd: since the same objection must for ever ly against all causes, except* primary *ones; which are certainly removed far beyond the reach of human inquiry. The proper office, and highest boast of true philosophy, is, to bring us still nearer to the* DEITY, *by leading us upwards, step by step, in the mighty* scale *of Nature.*—THOMAS MELVILL, "Observations on Light and Colours," Essays and Observations, Physical and Literary (Philosophical Society, Edinburgh, 1756), Vol. II, pp. 12–90. (Contributed by E. C. Watson.)

Computation of the Moment of Inertia of Various Bodies by the Approximate Sum of Numerical Series

EDDIE ORTIZ MUÑIZ
College of Agriculture and Mechanic Arts, Mayaguez, Puerto Rico
(Received March 12, 1952)

The paper presents a simple method of computing moments of inertia of various bodies by approximating the sum of numerical series. The method lends itself to be used in a course of college physics where most students are not familiar with calculus.

IN the analysis of problems concerning the mechanics of rigid bodies, the expression $I=\sum mr^2$, where m is the mass of a particle of the body at a distance r from the axis of rotation, occurs frequently, notably in determining the behavior of a rotating body. We might appropriately name $\sum mr^2$ the rotational inertia.

In determining the moment of inertia of a body with respect to a line it is always possible to select elements of mass so that:

1. All points in each element of mass are equally distant from the axis with respect to which the moment of inertia is to be found.

2. The moment of inertia of the element, with respect to the axis about which the moment of inertia of the whole body is to be found, is known; the moment of inertia of the body is found by summing up the moments of inertia of the elements.

3. The centroid of the elements is known and also the moment of inertia of the element with respect to an axis which passes through the centroid of the element and is parallel to the given axis; the moment of inertia of the element may then be expressed by means of the parallel axes theorem, that is

$$I=\bar{I}+md^2,$$

where I, $\bar{I}$, m, and d are, respectively, the moment of inertia of the element about the axis under consideration, the moment of inertia with respect to the centroid, the mass of the element, and the distance between the two parallel axes. The moment of inertia is then found by summing the moments of inertia of the different elements.

The purpose of this paper is to present a simple method of computing the moments of inertia of various bodies by approximating the sum of numerical series. It is hoped that the method will be found useful to students not familiar with calculus.

NUMERICAL SERIES INVOLVED

Throughout this paper the body under consideration will be divided into a large finite number of units. As the size of the unit is arbitrary, as small as one pleases, and hence the number of units very large, the sum of the necessary arithmetical series can be approximated as shown below:

Series	Exact value	Approximate value	
$\sum_1^N n$	$\frac{N(N+1)}{2}$	$\frac{N^2}{2}$	(1)
$\sum_1^N n^2$	$\frac{N(N+1)(2N+1)}{6}$	$\frac{N^3}{3}$	(2)
$\sum_1^N n^3$	$\frac{N^2(N+1)^2}{4}$	$\frac{N^4}{4}$	(3)
$\sum_1^N n^4$	$\frac{N(N+1)(2N+1)[3N(N+1)-1]}{30}$	$\frac{N^5}{5}$.	(4)

MOMENT OF INERTIA OF VARIOUS BODIES

1. Moment of inertia of a triangular plate about its base. Suppose the height h of the triangle (Fig. 1) is divided into h units; then the triangle is built up by laying side by side h rectangles having unit height.

The moment of inertia of the shaded portion n units from the base is $\Delta I=\sigma a n^2$, where $a=b(h-n)/h$, and σ is the mass per unit area.

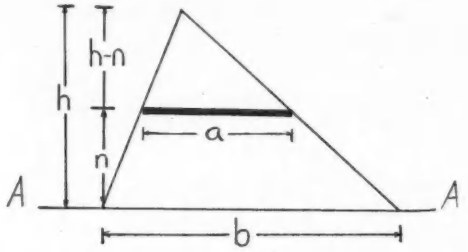


FIG. 1. Moment of inertia of a triangular plate about its base. The mass element is selected according to the first rule.

The moment of inertia of the triangular plate about AA is

$$I=\sum_{1}^{h}\frac{\sigma b(h-n)n^2}{h}=\frac{\sigma b}{h}[h\sum_{1}^{h}n^2-\sum_{1}^{h}n^3].$$

Using the approximate value from series (2) and (3),

$$I=\frac{\sigma b}{h}\left(\frac{h^4}{3}-\frac{h^4}{4}\right)=\frac{\sigma bh^3}{12}.$$

As the mass of the triangle is $M=\sigma bh/2$, then $I=Mh^2/6$.

2. Moment of inertia of a rod about an axis normal to the length at the center. Suppose the rod is divided into L units each of mass m (Fig. 2), then the moment of inertia of the mass n units from the axis is $\Delta I=mn^2$. The moment of inertia of the rod about AA is

$$I=2m\sum_{1}^{L/2}n^2.$$

Using the approximate value for series (2),

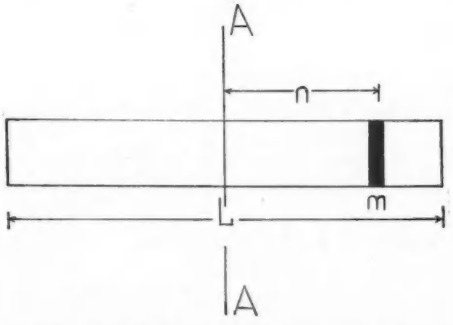


FIG. 2. Moment of inertia of a rod about an axis normal to the length at the center.

$I=mL^3/12$. Since the mass of the rod is $M=mL$, then $I=ML^2/12$.

3. Moment of inertia of a rectangular plate about an axis normal to the plate at the center. Suppose the length b of the rectangular plate is divided into b units (Fig. 3); then the rectangle is built up by laying side by side b rods having unit width. As the moment of inertia of a rod about an axis normal to the length at the center is known, applying the parallel axes theorem, the moment of inertia of the shaded rod n units from the axis under consideration is $\Delta I=\sigma an^2+\sigma a^3/12$, where σ is the mass per unit area.

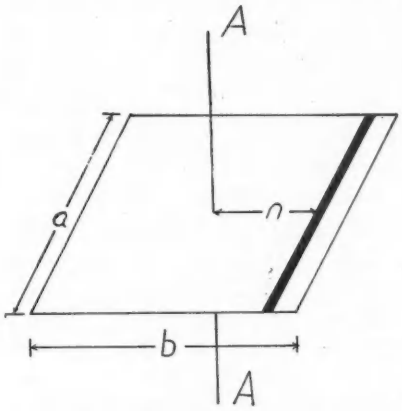


FIG. 3. Moment of inertia of a rectangular plate about an axis normal to the plate at the center. The mass element is selected according to the third rule.

The moment of inertia of the rectangular plate about AA is

$$I=2\sigma a\sum_{1}^{b/2}\left(n^2+\frac{a^2}{12}\right)=2\sigma a\left(\frac{ba^2}{24}+\sum_{1}^{b/2}n^2\right).$$

Using the approximate value for series (2),

$$I=2\sigma a\left(\frac{ba^2}{24}+\frac{b^3}{24}\right)=\frac{\sigma ba}{12}(a^2+b^2).$$

Since the mass of the rectangle is $M=\sigma ab$, then

$$I=\frac{M}{12}(a^2+b^2).$$

4. Moment of inertia of a disk about an axis normal to the plane of the disk at the center. A body that has all its mass disposed at equal dis-

tances from its axis of rotation (for example, a circular ring) has a rotational inertia that equals its total mass times the square of its radius.

Suppose the disk is divided into R concentric rings each having unit thickness (Fig. 4); then the moment of inertia of the shaded ring having a radius of n units about an axis normal to the plane of the ring is $\Delta I = 2\pi\sigma n^3$, where σ is the mass per unit area. The moment of inertia of the disk about AA is

$$I = 2\pi\sigma \sum_1^R n^3.$$

Using the approximate value to series (3),

$$I = \pi\sigma R^4/2.$$

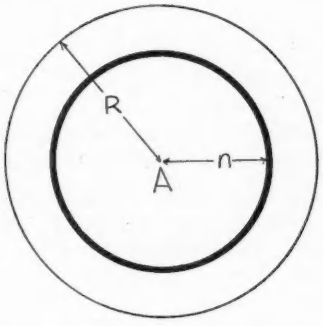


FIG. 4. Moment of inertia of a disk about an axis normal to the plane of the disk at the center. The element of mass is selected according to the first rule.

As the mass of the disk is $M = \pi\sigma R^2$, therefore,

$$I = MR^2/2.$$

5. Moment of inertia of a right circular cone about its geometrical axis. Suppose the height h of the cone is divided into h units (Fig. 5); then the cone is built up by laying one over the other h thin disks having unit thickness.

The moment of inertia of the shaded thin disk about the axis is $\Delta I = \pi\rho a^4/2$, where ρ is the mass per unit volume and $a = R(h-n)/h$. The moment of inertia of the cone about AA is

$$I = \frac{\pi\rho R^4}{2} \sum_1^h \left(1 - \frac{n}{h}\right)^4$$

$$= \frac{\pi\rho R^4}{2} \sum_1^h \left(1 - \frac{4n}{h} + \frac{6n}{h^2} - \frac{4n^3}{h^3} + \frac{n^4}{h^4}\right).$$

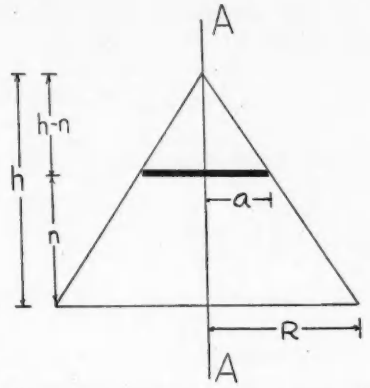


FIG. 5. Moment of inertia of a right circular cone about its geometrical axis. The element of mass is selected according to the second rule.

Using the approximate values of series (1), (2), (3), and (4),

$$I = \frac{\pi\rho R^4}{2}\left(h - 2h + 2h - h + \frac{h}{5}\right) = \frac{\pi\rho h R^4}{10}.$$

Since the mass of the cone is $M = \pi\rho h R^2/3$, then $I = 3MR^2/10$.

6. Moment of inertia of a sphere about any diameter. Suppose the radius R of the sphere is divided into R units (Fig. 6); then the sphere is built up by laying one over the other $2R$ thin disks having unit thickness.

The moment of inertia of the shaded thin disk about the axis is $\Delta I = \pi\rho r^4/2$, where ρ is the

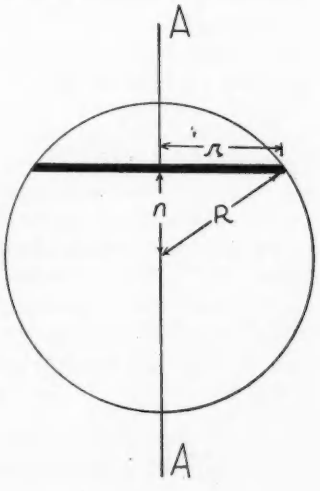


FIG. 6. Moment of inertia of a sphere about any diameter. The element of mass is selected according to the second rule.

mass per unit volume and $r=\sqrt{(R^2-n^2)}$. The moment of inertia of the sphere about AA is

$$I=\pi\rho\sum_1^R(R^2-n^2)^2=\pi\rho\sum_1^R(R^4-2R^2n^2+n^4).$$

Using the approximate values of series (2) and (4),

$$I=\pi\rho\left(R^5-\frac{2R^5}{3}+\frac{R^5}{5}\right)=\frac{8\pi\rho R^5}{15}.$$

Since the mass of the sphere is $M=4\pi\rho R^3/3$, then $I=2MR^2/5$.

It has thus been shown that for a given body the moment of inertia can be expressed in terms of the sum of a numerical series. The approximate sum of the series being known, the moment of inertia can be found at once. It is obvious that the method could very well be applied to computation of center of mass. Because of its simplicity the method lends itself to be used in a course of college physics and in the light of the author's experience it may prove to be a valuable teaching aid.

It is a pleasure to express my thanks to Professor Braulio Dueño Leary, Professor Jose Medina Hernandez, and Professor Jorge Rodriguez Arias for having given valuable suggestions and for reading the paper.

Achievement of Physics Students With and Without Laboratory Work*

HAYM KRUGLAK
University of Minnesota, Minneapolis 14, Minnesota

(Received June 9, 1952)

Two general physics laboratory groups were compared in achievement with a group enrolled in the same course without laboratory. The achievement was measured by two pencil-paper examinations and two laboratory performance tests. The analysis of variance and covariance method was used to hold constant three measures of initial differences among students. There were no statistically significant differences between the means of the groups on the mechanics theory test. The laboratory groups were significantly better than the no-laboratory group on all tests dealing with laboratory work.

THE PROBLEM

AN experiment on the relative effectiveness of the conventional and demonstration laboratory methods was described in a previous paper.[1] The analysis has been extended so as to include the data on a group of students who were not exposed to laboratory experiences. The present investigation was designed to test two null hypotheses: (1) There are no differences in the learning outcomes of general physics students who do laboratory work by the individual method and students who take the same course *without* laboratory. (2) There are no differences in the learning outcomes of general physics students to whom laboratory experiments are demonstrated and students who take the same course *without* laboratory.

EXPERIMENTAL PROCEDURE

The 28 students in the individual or conventional laboratories and the 28 students in the demonstration laboratories were enrolled in Physics 1a, during the fall quarter 1950, at the University of Minnesota. The characteristics of the course population, the method of sampling, and the contrasting treatment in the laboratory were described in the above-mentioned paper.

The no-laboratory group consisted of 21 male *volunteers*, enrolled in Physics 1 during the same quarter as the two laboratory groups. Physics 1 and Physics 1a have the same prerequisites, cover the same subject matter, and have the same lecturer. However, students in Physics 1a take two hours of laboratory per week for one additional quarter-credit, while students in Physics 1 take no laboratory at all.

Since randomization is one of the basic assumptions in the analysis of variance and co-

* This work was supported n part by the joint program of the AEC and ONR.

[1] H. Kruglak, Am. J. Phys. 20, 136 (1952).

TABLE I. Summary of the instructional outcomes. Physics 1, 1a, fall quarter 1950, University of Minnesota.

Criterion	Laboratory method[a]	N	Max. possible score	Adjusted means[b]	Significance of differences between means
Mechanics theory	C	28	132	58.05	(C−D) Not significant
	D	28		54.48	(C−N) Not significant
	N	21		50.80	(D−N) Not significant
					(C+D)−N Not significant
Laboratory Written	C	28	120	42.07	(D−C) Not significant
	D	28		44.20	(C−N) Significant at the 5 percent level
	N	21		30.49	(D−N) Significant at the 1 percent level
					(C+D)−N Significant at the 1 percent level
Short-item laboratory performance	C	28	180	90.51	(C−D) Significant at the 1 percent level
	D	28		72.31	(C−N) Significant at the 1 percent level
	N	21		56.81	(D−N) Significant at the 1 percent level
					(C+D)−N Significant at the 1 percent level
Long-item laboratory performance	C	28	180	78.68	(C−D) Not significant
	D	28		73.62	(C−N) Significant at the 1 percent level
	N	21		52.03	(D−N) Significant at the 1 percent level
					(C+D)−N Significant at the 1 percent level

[a] C = conventional or individual,
D = demonstration,
N = no laboratory.
[b] Adjusted by analysis of variance and covariance for initial differences in scores on the American Council on Education Psychological Examination, mathematics pre-test, and criterion pre-test.

variance, the sample of volunteers was compared to the remainder of the male population in Physics 1. No significant differences between the means of the volunteers and nonvolunteers were found on the ACE, the mathematics pre-test, the mechanics theory pre-test, and the laboratory written pre-test. The volunteer group was adjudged to be representative of the course population.

The three groups in this study were homogeneous in that every student was a graduate of an accredited Minnesota high school, was enrolled in the course for credit, had a score on a scholastic aptitude test (ACE), took all the special pre-tests and post-tests, and completed the quarter's work in physics.

The control factors for the three groups were (1) identical lectures, (2) identical textbook and assignments, (3) identical course quizzes, and (4) identical pre-tests and post-tests.

The students in the conventional groups worked in pairs, assembling and operating the apparatus as specified in the laboratory manual. The laboratory teaching assistants performed all the experiments for the demonstration groups, without any manipulation on the part of the students. The no-laboratory group had no laboratory instruction during the quarter.

The achievement of each group was measured by means of a mechanics theory test, a laboratory paper-pencil test, and two laboratory performance examinations.[1]

The volunteers were given a ten percent increase in their final grades as compensation for taking part in the experiment. The volunteer group appeared to be well-motivated and serious-minded during the test periods.

ANALYSIS OF DATA

The analysis of variance and covariance techniques were used to test the hypotheses of no difference between the means of the groups on the four criterion tests. The analysis of covariance made it possible to hold constant the student's scores on the ACE, the mathematics pre-test, and the pre-test corresponding to the criterion under consideration. The summary of the analysis is shown in Table I.

Testing the null hypotheses by the F test gave the following results:

(a) The hypothesis of no difference between the individual laboratory group and the no-laboratory group for the

(1) mechanics theory—accepted,
(2) laboratory written—rejected,

(3) short-item laboratory performance—rejected,
(4) long-item laboratory performance—rejected.

(b) The hypothesis of no difference between the demonstration laboratory group and the no-laboratory group for the

(1) mechanics theory—accepted,
(2) laboratory written—rejected,
(3) short-item laboratory performance—rejected,
(4) long-item laboratory performance—rejected.

Further inspection of the table shows that on three of the tests the adjusted means are in descending order, with the conventional group at the top. On all of the criteria the no-laboratory group had the lowest means.

CONCLUSIONS

The experimental evidence of this investigation supports the conclusion that students who get laboratory instruction, by the individual *or* demonstration method, are superior to students without such instruction, on tests designed to measure *laboratory* outcomes.

Since no significant differences between the means of any two groups were found for the mechanics theory test, it is reasonable to conclude that laboratory instruction does not significantly influence scores on pencil-paper tests designed to measure a knowledge of facts, principles, and applications of elementary mechanics.

The difference between the means of the individual and the demonstration groups on the short-item performance test is about equal to the difference between the means of the demonstration and the no-laboratory groups on the same test. This fact might be used as evidence in support of the "learning by doing" pedagogical concept, since the short-item performance test attempts to evaluate simple manipulatory skills, measuring techniques, and apparatus knowledge.

RECOMMENDATIONS

On the basis of this investigation it is recommended that the evaluation of laboratory instruction be made in terms of the objectives of laboratory work.

It is recommended that pencil-paper *and* performance tests be used to measure the general and specific outcomes of laboratory instruction.

Since there were no significant differences between the means of the individual and the demonstration groups on the problem-solving (long-item) performance test, it is recommended that more time be spent in the laboratory on problematic situations.

The evaluation of laboratory teaching is of considerable interest and importance. Although a great deal of staff time, apparatus, and building space is given over to instructional laboratories, there have been relatively few investigations dealing with measurable outcomes of laboratory instruction at the freshman and sophomore levels. It is recommended and hoped that similar studies will be carried out in other schools, with nonscience majors as well as technical students, with other methods and tests, and for longer instructional periods.

The writer recognizes that all the conclusions and recommendations of this study are limited by the population used in the three groups, by the design of the experiment, and by the four measures of instructional outcomes.

It is a pleasure to acknowledge the invaluable help of Mr. W. J. Moonan in planning the statistical treatment. Thanks are due to Mrs. N. Horwitz for carrying out most of the computations. The encouragement and advice of Professor C. N. Wall are greatly appreciated.

There have been lean seasons for adventurers before, the eighteenth century notably, when everything seemed owned, done, mapped. In such times the new is to be sought inwards, not in immutable Nature, but in the ever renewed flux of human life.—WILLIAM BOLITHO, Twelve Against the Gods.

Determination of Molecular Structure from Microwave Spectroscopic Data

J. KRAITCHMAN*
Westinghouse Research Laboratories, East Pittsburgh, Pennsylvania
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A method is described for determining the position of an atom in a molecule from spectroscopic measurements on two isotopic species of the molecule. The method is applied to various types of molecules; explicit expressions are derived for linear, symmetric top, planar, and nonplanar asymmetric top molecules. The number of isotopic species on which measurements must be made to complete the structural determination, i.e., determine the position of every atom in the molecule, is discussed for various types of molecules. An application of the method to the determination of mass difference ratios is also considered.

I. INTRODUCTION

DETERMINATION of the size and shape of molecules from spectroscopic measurements in the microwave region has now become very common. Once the major work of identifying the spectrum of the molecule has been completed, it is possible to calculate rotational constants for the molecule, and from these the molecular moments of inertia. Except for diatomic molecules and a few asymmetric top molecules, e.g., NH_2D, SO_2, etc., the moments of inertia of one isotopic species of a molecule are not sufficient to determine the molecular configuration. However, when sufficient information is available on the moments of inertia of other isotopic species of the molecule, it becomes possible to determine the molecular configuration completely. In this paper a general and quite straightforward method of using this information to determine structural parameters is discussed. In particular, it is found that simple but exact expressions can be derived for determining the position of any atom in a molecule.

Usually the spectroscopist is satisfied if he can determine the equilibrium structural parameters of the molecule, i.e., the positions the atoms would occupy if they were at rest in the molecule. However, all spectroscopic measurements must be made on vibrating molecules, since even in the lowest or ground state the atoms have a certain amount of vibrational energy—the so-called zero-point energy. Consider, for example, a diatomic molecule in which the distance between the atoms is r. If the vibrations were harmonic, the average distance between the atoms during a vibration r_{Av} would be equal to the equilibrium distance r_e. Thus, the equilibrium moment of inertia would be proportional to r_e^2 or to $(r_{Av})^2$. However, the average moment of inertia for the vibration is proportional to the average square of the distance between the atoms, $(r^2)_{Av}$. Since $(r^2)_{Av}$ is not equal to $(r_{Av})^2$, the average moment of inertia is not equal to the equilibrium moment of inertia. Hence, even if it were possible to determine the average moment of inertia, the positions computed from it would not correspond to the equilibrium positions.

Actually, however, the analysis of a rotational spectrum gives not the average moments of inertia, but rather rotational constants, which are essentially reciprocals of the moments of inertia averaged over a vibrational state. The moments of inertia computed from these rotational constants are called effective moments of inertia. Thus, in a diatomic molecule the rotational constant is proportional to the average value of $1/r^2$, i.e., to $(1/r^2)_{Av}$. Since $(1/r^2)_{Av}$ is not equal to $1/(r_{Av})^2$, the effective moment of inertia will not be equal to the equilibrium moment of inertia. Consequently, positions computed from it will not correspond to equilibrium positions. (Since $(1/r^2)_{Av}$ is not equal to $1/(r^2)_{Av}$, the effective moment of inertia will also differ from the average moment of inertia.)

In some cases the average distance between the atoms during a vibration r_{Av} would be of greater interest than the equilibrium distance r_e. As mentioned above, if the vibrations were harmonic, these two quantities would be equal. However, to complicate matters further the

* Present address: Department of Chemistry, Columbia University, New York 27, New York.

vibrations are actually anharmonic, so that r_M is not equal to r_e. Since it usually has been impossible even to calculate equilibrium parameters, little or no attention has been devoted to the problem of determining average distances.

In the theory it is always possible to obtain equilibrium parameters. By determining the rotational constants of the molecule in a number of different vibrational states, it is possible to compute the equilibrium rotational constants.[1] From these the equilibrium moments of inertia and the equilibrium positions of the atoms can be determined. However, except for a few simple molecules, it has been impossible to measure the rotational constants in a sufficient number of different vibrational states to make this possible. In fact, in the vast majority of cases the microwave spectroscopist has been only able to determine the rotational constants in the ground state.

Of course, the spectroscopist could resign himself to using just effective moments of inertia. However, when it becomes necessary to use isotopic species of a molecule to complete the structural determination, the implicit assumption is made that the structural parameters are the same in all isotopic species of the molecule. Actually this assumption is only valid for the equilibrium structural parameters. Hence, if only to obtain a set of self-consistent parameters, equilibrium moments must be used. If for lack of sufficient information the spectroscopist is forced to use effective moments, not only will the calculated structural parameters differ from the equilibrium parameters, but they may not even exactly fit all the isotopic species measured.

These factors, which are responsible for the differences between calculated and equilibrium parameters, are known as the zero-point vibration effects; it is these zero-point effects that are responsible for the largest uncertainties in structural parameters determined from microwave data.

II. PRELIMINARY REMARKS

The inertia dyadic can be defined as[2]

$$\mathbf{I}=\sum_i m_i(r_i^2\mathbf{1}-\mathbf{r}_i\mathbf{r}_i), \tag{1}$$

where m_i is the mass of the ith atom, $\mathbf{r}_i$ the distance of the ith atom from the origin, and $\mathbf{1}$ the unit dyadic.[3] The diagonal elements of $\mathbf{I}$ are known as the axial moments of inertia or, simply, the moments of inertia, a typical one having the form

$$I_{zz}=\sum_i m_i(y_i^2+z_i^2). \tag{2a}$$

The nondiagonal elements are designated products of inertia[4] and have the form illustrated by

$$I_{xy}=I_{yx}=-\sum_i m_i x_i y_i. \tag{2b}$$

In matrix notation the inertia dyadic can be written

$$\mathbf{I}=\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix}.$$

It is always possible to rotate the coordinate axes so that this matrix is diagonal. In the language of matrix algebra such a diagonalization can be obtained by means of a similarity transformation. The rotated axes are called the principal axes, and the corresponding diagonal elements, I_X, I_Y, I_Z, are known as the principal moments of inertia. In practice the principal moments of inertia, being eigenvalues of the inertia matrix, are obtained as the three roots of the secular equation

$$\begin{vmatrix} I_{xx}-I & I_{xy} & I_{xz} \\ I_{xy} & I_{yy}-I & I_{yz} \\ I_{xz} & I_{yz} & I_{zz}-I \end{vmatrix}=0.$$

Spectroscopic measurements only give information as to the values of the principal moments of inertia with respect to axes passing through the center of mass of the molecule. If the origin of the coordinate axes is chosen at the center of mass of the molecule and Eq. (2) is used to compute the elements of the inertia dyadic, the diagonalization procedure will automatically give the principal moments of inertia with respect to the center of mass. However, it is often desirable to start with an arbitrary origin for the coordinate system and still make

[1] For details, see G. Herzberg, *Infrared and Raman Spectra of Polyatomic Molecules* (D. Van Nostrand Company, Inc., New York, 1945).

[2] See, for example, H. Goldstein, *Classical Mechanics* (Addison-Wesley Press Inc., Cambridge, 1950), Chap. 5.

[3] Certain very small errors are introduced by assuming that the nucleus of an atom and its associated electrons are located at a point. However, these errors are negligible compared to the precision to which the moments of inertia can be determined.

[4] Often their negatives are called the products of inertia.

the computations with respect to the center of mass. This can easily be done by making use of the parallel axis theorem. Let $\mathbf{r}_i$ represent the distance of a particular atom from the arbitrary origin, $\mathbf{r}_i^o$ the distance of this atom from the center of mass, and $\mathbf{r}$ the distance of the center of mass from the origin. Then

$$\mathbf{r}_i=\mathbf{r}_i^o+\mathbf{r},$$

and by definition of the center of mass

$$\mathbf{r}=\sum_i m_i\mathbf{r}_i/\sum_i m_i, \quad \sum_i m_i\mathbf{r}_i^o=0.$$

Upon substitution of these relations into Eq. (1), the following expression is obtained:

$$\mathbf{I}^o=\sum_i m_i(r_i^2\mathbf{1}-\mathbf{r}_i\mathbf{r}_i)-(1/\sum_i m_i)[(\sum_i m_i\mathbf{r}_i)\cdot(\sum_i m_i\mathbf{r}_i)\mathbf{1}-(\sum_i m_i\mathbf{r}_i)(\sum_i m_i\mathbf{r}_i)], \quad (3)$$

where $\mathbf{I}^o$ is the inertia dyadic with respect to the center of mass. Typical elements have the form[5]

$$I_{xx}^o=\sum_i m_i(y_i^2+z_i^2)-\frac{(\sum_i m_i y_i)^2}{\sum_i m_i}-\frac{(\sum_i m_i z_i)^2}{\sum_i m_i}, \quad (4a)$$

$$I_{xy}^o=-\sum_i m_i x_i y_i+\frac{(\sum_i m_i x_i)(\sum_i m_i y_i)}{\sum_i m_i}. \quad (4b)$$

With an arbitrary origin for the coordinate axes the diagonalization procedure will automatically give the principal moments of inertia with respect to the center of mass if the elements of the inertia dyadic are computed from Eq. (4). Note, that if the arbitrary origin is the center of mass, Eq. (4) reduces to Eq. (2). In the future, since all moments of inertia will be computed with respect to the center of mass, Eq. (4) will always be used; for simplicity the superscript *o* will be omitted.

In dealing with asymmetric top molecules it will be more convenient to use another symmetric dyadic defined by

$$\mathbf{P}=\sum_i m_i\mathbf{r}_i\mathbf{r}_i-\frac{(\sum_i m_i\mathbf{r}_i)(\sum_i m_i\mathbf{r}_i)}{\sum_i m_i}, \quad (5)$$

where $\mathbf{P}$ is the planar dyadic with respect to the center of mass. If the origin of the coordinate axes is at the center of mass, $\mathbf{P}$ reduces to

$$\mathbf{P}=\sum_i m_i\mathbf{r}_i\mathbf{r}_i.$$

Typical elements of $\mathbf{P}$ have the form

$$P_{xx}=\sum_i m_i x_i^2-[(\sum_i m_i x_i)^2/\sum_i m_i], \quad (6a)$$

$$P_{xy}=\sum_i m_i x_i y_i-[(\sum_i m_i x_i)(\sum_i m_i y_i)/\sum_i m_i], \quad (6b)$$

which, when the origin of the coordinates axes is at the center of mass, reduce to

$$P_{xx}=\sum_i m_i x_i^2, \quad (7a)$$

$$P_{xy}=\sum_i m_i x_i y_i. \quad (7b)$$

The diagonal elements of $\mathbf{P}$ are often called planar moments of inertia. The nondiagonal elements, except in choice of sign, are equivalent to the products of inertia.

Relations between the elements of $\mathbf{P}$ and $\mathbf{I}$ can readily be derived from the expressions given above. In this paper only the relations between the principal moments will be needed; the following are typical of these:

$$P_X=\tfrac{1}{2}(I_Y+I_Z-I_X), \quad (8)$$

$$I_X=P_Y+P_Z. \quad (9)$$

III. GENERAL METHOD

Consider a molecule of mass M for which the principal moments of inertia have been found to be I_X, I_Y, I_Z. Let a coordinate system be set up with the origin at the center of mass of the molecule and with axes X, Y, Z oriented in the directions of the principal moments. Since

$$\sum_i m_i x_i=0, \quad \sum_i m_i y_i=0, \quad \sum_i m_i z_i=0, \quad (10a)$$

the elements of $\mathbf{I}$ given by Eq. (4) reduce to

$$\begin{aligned} I_{xy}&=-\sum_i m_i x_i y_i=0,\\ I_{xz}&=-\sum_i m_i x_i z_i=0,\\ I_{yz}&=-\sum_i m_i y_i z_i=0, \end{aligned} \quad (10b)$$

$$\begin{aligned} I_X&=I_{xx}=\sum_i m_i(y_i^2+z_i^2),\\ I_Y&=I_{yy}=\sum_i m_i(x_i^2+z_i^2),\\ I_Z&=I_{zz}=\sum_i m_i(x_i^2+y_i^2). \end{aligned} \quad (10c)$$

The nine Eqs. (10) will be referred to as the moment equations of the original molecule.

[5] The usefulness of these relations in computing moments and products of inertia has previously been pointed out by J. O. Hirschfelder, J. Chem. Phys. 8, 431 (1940).

Let an atom of mass m in the original molecule be replaced by an isotope of mass $m+\Delta m$; denote the coordinates of this atom with respect to the center of mass of the original molecule by x, y, z. It is then convenient to compute the elements of **I** for the isotopically substituted molecule using the coordinate system set up for the original molecule. Even if this is done, by using Eq. (4), the elements of **I** are given with respect to the center of mass of the substituted molecule. It is then found that for the substituted molecule

$$I_{zz}'=I_X+\Delta m(y^2+z^2)-\frac{(\Delta my)^2}{M+\Delta m}-\frac{(\Delta mz)^2}{M+\Delta m}$$
$$=I_X+\mu(y^2+z^2), \quad (11)$$
$$I_{xy}'=-\mu xy, \text{ etc.},$$

where $\mu=M\Delta m/M+\Delta m$, and M is the mass of the original molecule. From spectroscopic measurements the principal moments of inertia of the substituted molecule, I_X', I_Y', I_Z', can be determined. The problem, then, is to compute x, y, z in terms of I_X, I_Y, I_Z and I_X', I_Y', I_Z'.

In general, the principal axes X', Y', Z' of the substituted molecule will be rotated with respect to the principal axes X, Y, Z of the original molecule. However, for many simpler molecules at least one of the principal axes of the substituted molecule will be parallel to a principal axis of the original molecule. This special condition will be met whenever one of the principal axes of the original molecule is perpendicular to the line joining the center of mass and the atom to be isotopically substituted. If this perpendicular principal axis is the Z axis, then the Z' axis will be parallel to the Z axis and the principal moments of the original and substituted molecules will obey the relation

$$I_Z'-I_Z=I_X'-I_X+I_Y'-I_Y. \quad (12)$$

Since to date most investigations in microwave spectroscopy have involved molecules which meet this condition, such molecules will be considered first. The more general case will be treated in Sec. VII.

IV. LINEAR MOLECULES

Let the Z axis be oriented along the molecular axis. Then $I_X=I_Y$, and $I_Z=0$. With this choice of axes the X' axis can be chosen parallel to the X axis and the Y' axis parallel to the Y axis; the Z' axis will coincide with the Z axis. Since the x and y coordinates of all atoms in the molecule are zero, Eq. (11) reduces to

$$I_{xy}'=I_{xz}'=I_{yz}'=0,$$
$$I_X'=I_X+\mu z^2,$$
$$I_Y'=I_Y+\mu z^2,$$
$$I_Z'=I_Z=0.$$

Thus, the absolute value of the distance of the substituted atom from the center of mass of the original molecule is

$$|z|=[\mu^{-1}(I_Y'-I_Y)]^{\frac{1}{2}}=[\mu^{-1}(I_X'-I_X)]^{\frac{1}{2}}. \quad (13)$$

V. SYMMETRIC TOP MOLECULES

Let the Z axis be oriented along the symmetry axis of the molecule. Then $I_X=I_Y$. Consider the following types of substitutions:

(A) Substitution on an Atom Located on the Symmetry Axis

With the choice of axes given above the X' axis can be chosen parallel to the X axis and the Y' axis parallel to the Y axis; the Z' axis will coincide with the Z axis. The x and y coordinates of any atom located on the symmetry axis are zero. In all the mathematical details this case is exactly similar to the linear molecule treated above, except that I_Z and I_Z', while equal, are not zero. However, since the microwave spectrum of a symmetric top gives no information as to the magnitude of I_Z or I_Z', this distinction is immaterial to the treatment being given here. Thus, the distance of a substituted atom, located on the symmetry axis, from the center of mass of the original molecule can be determined from Eq. (13).

(B) Substitution on an Atom not Located on the Symmetry Axis[6]

In this case it is always possible to orient either the X or Y axis so that one is perpendicular to the line joining the center of mass and the atom to be substituted. Let the axes be so oriented that the X axis is this perpendicular axis. With this orientation the x coordinate of the

[6] While the original molecule is a symmetric top, the isotopically substituted molecule is not.

substituted atom is zero. Equation (11) becomes

$$I_{xy}'=I_{xz}'=0,$$
$$I_{yz}'=-\mu yz,$$
$$I_X'=I_X+\mu(y^2+z^2),$$
$$I_{yy}'=I_Y+\mu z^2,$$
$$I_{zz}'=I_Z+\mu y^2.$$

The third of these equations immediately gives

$$y^2+z^2=\mu^{-1}(I_X'-I_X). \tag{14a}$$

Furthermore, the inertia matrix of the substituted molecule can be reduced to the form

$$\begin{pmatrix} I_Y+\mu z^2 & -\mu yz \\ -\mu yz & I_Z+\mu y^2 \end{pmatrix}.$$

Since the trace of a matrix is invariant under a similarity transformation, it is possible to write

$$I_Y+\mu z^2+I_Z+\mu y^2=I_Y'+I_Z',$$

or

$$y^2+z^2=\mu^{-1}(I_Y'-I_Y+I_Z'-I_Z). \tag{14b}$$

Comparison of this expression with Eq. (14a) gives

$$I_X'-I_X=I_Y'-I_Y+I_Z'-I_Z.$$

Except for a slight change in notation (since the X axis is the perpendicular axis in this case) this expression is equivalent to Eq. (12). Furthermore, since $I_X=I_Y$, this relation reduces to

$$I_Z=I_Z'+I_Y'-I_X'. \tag{15}$$

It is also a well-known theorem of matrix algebra that the determinant of a matrix is invariant under a similarity transformation. This gives the equation

$$(I_Y+\mu z^2)(I_Z+\mu y^2)-\mu^2y^2z^2=I_Y'I_Z'. \tag{16}$$

Equations (14b) and (16) can now be solved explicitly for y and z. The results are

$$|y|=\left[\frac{1}{\mu}\frac{(I_Z'-I_Z)(I_Y'-I_Z)}{(I_Y-I_Z)}\right]^{\frac{1}{2}}$$
$$=\left[\frac{1}{\mu}(I_Z'-I_Z)\left(1+\frac{I_Y'-I_Y}{I_Y-I_Z}\right)\right]^{\frac{1}{2}}, \tag{17a}$$

$$|z|=\left[\frac{1}{\mu}\frac{(I_Y'-I_Y)(I_Z'-I_Y)}{(I_Z-I_Y)}\right]^{\frac{1}{2}}$$
$$=\left[\frac{1}{\mu}(I_Y'-I_Y)\left(1+\frac{I_Z'-I_Z}{I_Z-I_Y}\right)\right]^{\frac{1}{2}}. \tag{17b}$$

The moment of inertia about the symmetry axis of the original molecule I_Z can not be evaluated from the rotational spectrum. However, I_Z can be eliminated from these expressions using Eq. (15), giving

$$|y|=\left[\frac{1}{\mu}\frac{(I_X'-I_Y')(I_X'-I_Z')}{(I_X'-I_Y'-I_Z'+I_Y)}\right]^{\frac{1}{2}}, \tag{18a}$$

$$|z|=\left[\frac{1}{\mu}\frac{(I_Y'-I_Y)(I_Z'-I_Y)}{(I_Y'+I_Z'-I_X'-I_Y)}\right]^{\frac{1}{2}}. \tag{18b}$$

Thus, the y and z coordinates of the substituted atom can be determined from Eq. (18);[7] the distance of the substituted atom from the center of mass can be obtained from Eq. (14). It is also possible to calculate the moment of inertia of the original molecule about its symmetry axis from Eq. (15).

VI. PLANAR ASYMMETRIC TOP MOLECULES

Let the Z axis be chosen perpendicular to the plane of the molecule. Then $I_X+I_Y=I_Z$. With this choice of axes the Z' axis will be parallel to the Z axis, and the z coordinates of all atoms zero. Equation (11) reduces to

$$I_{xz}'=I_{yz}'=0,$$
$$I_{xy}'=-\mu xy,$$
$$I_{xx}'=I_X+\mu y^2,$$
$$I_{yy}'=I_Y+\mu x^2,$$
$$I_Z'=I_Z+\mu(x^2+y^2).$$

In its mathematical details the treatment here is similar to that given for symmetric tops in Sec. V(B). Equation (14) becomes

$$x^2+y^2=\mu^{-1}(I_Z'-I_Z), \tag{19}$$

while Eq. (17) becomes

$$|x|=\left[\frac{1}{\mu}\frac{(I_Y'-I_Y)(I_X'-I_Y)}{(I_X-I_Y)}\right]^{\frac{1}{2}}$$
$$=\left[\frac{1}{\mu}(I_Y'-I_Y)\left(1+\frac{I_X'-I_X}{I_X-I_Y}\right)\right]^{\frac{1}{2}}, \tag{20a}$$

$$|y|=\left[\frac{1}{\mu}\frac{(I_X'-I_X)(I_Y'-I_X)}{(I_Y-I_X)}\right]^{\frac{1}{2}}$$
$$=\left[\frac{1}{\mu}(I_X'-I_X)\left(1+\frac{I_Y'-I_Y}{I_Y-I_X}\right)\right]^{\frac{1}{2}}. \tag{20b}$$

[7] If the isotopically substituted molecule is only a very slightly asymmetric top, often I_Z' can not be evaluated. In this case, since Eq. (18) is inapplicable, an explicit general solution can not be given.

VII. NONPLANAR ASYMMETRIC TOPS

In this case it is much more convenient to use the planar dyadic **P** defined in Eq. (5). In terms of this dyadic the moment equations [Eqs. (10b) and (10c)] for the original molecule become

$$\begin{aligned} P_{xy}&=\textstyle\sum_i m_i x_i y_i=0,\\ P_{xz}&=\textstyle\sum_i m_i x_i z_i=0,\\ P_{yz}&=\textstyle\sum_i m_i y_i z_i=0, \end{aligned} \tag{21a}$$

$$\begin{aligned} P_X&=\textstyle\sum_i m_i x_i^2,\\ P_Y&=\textstyle\sum_i m_i y_i^2,\\ P_Z&=\textstyle\sum_i m_i z_i^2, \end{aligned} \tag{21b}$$

while for the substituted molecule the elements of **P** are of the form

$$\begin{aligned} P_{xx}'&=P_X+\mu x^2,\\ P_{xy}'&=\mu xy, \text{ etc.} \end{aligned} \tag{22}$$

The secular equation for the substituted molecule, of which the roots are the planar principal moments P_X', P_Y', P_Z', is

$$\begin{vmatrix} P_X+\mu x^2-P' & \mu xy & \mu xz \\ \mu xy & P_Y+\mu y^2-P' & \mu yz \\ \mu xz & \mu yz & P_Z+\mu z^2-P' \end{vmatrix}=0.$$

This secular equation can be expanded to give a cubic equation in P'. It is well known from the theory of equations that there are definite relations between the coefficients of a cubic equation and the roots of the equation. In particular, if P_X', P_Y', P_Z' are the roots of a cubic equation in P', the equation may be written

$$P'^3-(P_X'+P_Y'+P_Z')P'^2+(P_X'P_Y'+P_X'P_Z'+P_Y'P_Z')P'-P_X'P_Y'P_Z'=0.$$

By equating corresponding coefficients of the expanded secular equation to those in this expression, three equations in x^2, y^2, z^2 can be obtained. Solution of these three equations yields

$$\begin{aligned} |x|&=\left[\frac{1}{\mu}\frac{(P_X'-P_X)(P_Y'-P_X)(P_Z'-P_X)}{(P_Y-P_X)(P_Z-P_X)}\right]^{\frac{1}{2}}\\ &=\left[\frac{1}{\mu}(P_X'-P_X)\left(1+\frac{P_Y'-P_Y}{P_Y-P_X}\right)\right.\\ &\qquad\left.\times\left(1+\frac{P_Z'-P_Z}{P_Z-P_X}\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{23}$$

Analogous expressions for y and z can be obtained by cyclic permutation of the subscripts.[8] The distance of the substituted atom from the center of mass of the original molecule can be determined from the expression

$$|r|=[\mu^{-1}(P_X'-P_X+P_Y'-P_Y+P_Z'-P_Z)]^{\frac{1}{2}}. \tag{24}$$

Use of the relations given in Eq. (8) makes it possible to rewrite these expressions in terms of the axial moments of inertia

$$\begin{aligned} |x|=\left\{\frac{1}{2\mu}[(I_Y'-I_Y)+(I_Z'-I_Z)-(I_X'-I_X)]\right.\\ \times\left[1+\frac{(I_X'-I_X)-(I_Y'-I_Y)+(I_Z'-I_Z)}{2(I_X-I_Y)}\right]\\ \left.\times\left[1+\frac{(I_X'-I_X)+(I_Y'-I_Y)-(I_Z'-I_Z)}{2(I_X-I_Z)}\right]\right\}^{\frac{1}{2}}, \end{aligned} \tag{25}$$

$$|r|=[(2\mu)^{-1}(I_X'-I_X+I_Y'-I_Y+I_Z'-I_Z)]^{\frac{1}{2}}. \tag{26}$$

VIII. DETERMINATION OF MOLECULAR STRUCTURE

The parameter equations derived above enable the position of an atom to be determined with respect to the center of mass of the molecule. If the Cartesian coordinates of two atoms have been determined, it is a relatively simple problem in geometry to calculate the distance between them. Hence, when sufficient data are available to determine the positions of all atoms in the molecule, the bond distances and bond angles of the molecule can easily be calculated.

One isotopic substitution made on a particular atom is all that is required to determine the position of that atom. An additional substitution made on the same atom should not give any new information on the position of the atom. However, when effective moments of inertia are used instead of equilibrium values, the calculated structural parameters will be in error due to the neglect of zero-point vibrations. This will cause the parameters calculated from the additional

[8] D. K. Coles, *Advances in Electronics* (Academic Press, New York, 1950), Vol. II, p. 317, has given approximate expressions accurate to order $(\Delta m/M)^3$ for x, y, and z. Except for a numerical factor, his results can be shown to be in agreement with the exact expressions given here.

substitution to differ from those calculated from the first substitution. These differences are sometimes useful in estimating the magnitude of the zero-point vibration effects. Some attempts have been made to correct for the zero-point vibrations by correlating these differences with the change in mass with isotopic substitution.[9] As yet no completely satisfactory method for making such corrections has been developed.

Substitution on an atom near the center of mass of the molecule, which produces only a small change in the moment of inertia, may still cause a marked change in the vibrational frequencies and, hence, in the effects due to zero-point vibrations. It has even been found that in some cases the zero-point vibration effects are so large that, if neglected, imaginary values would be obtained for certain of the bond distances.[10] Consequently, it is always desirable to avoid substituting on an atom near the center of mass.

It is apparent from the above that if the molecule is assumed to be rigid, i.e., zero-point vibrations can be neglected, a single substitution on a particular atom suffices to determine the position of that atom. The problem of determining the minimum number of isotopic substitutions (on different atoms) necessary to completely determine the positions of all the atoms in the molecule is somewhat complicated. One complication arises from the fact that the parameter equations only give information as to the magnitude of the coordinates of the substituted atom, i.e., the signs of the coordinates are not determined. These ambiguities arise directly from the physical nature of the problem, *viz.*, moments of inertia depend on length squared. In general, by making additional substitutions it would be possible to resolve these ambiguities. However, a much more profitable approach is to make use of structural information available from other sources, e.g., infrared spectroscopy, electron diffraction measurements, theoretical chemistry, etc., to resolve them. In the ensuing discussion the attitude will be adopted that any ambiguities arising can be resolved by recourse to other sources of data. When this is not possible, it will be necessary to make additional substitutions above the minimum number.

Consider, then, an n atomic *linear* molecule in which $n-2$ isotopic substitutions have been made. If the molecular axis is taken to be along the Z axis, the magnitude of the z coordinates of the $n-2$ substituted atoms can be determined from the parameter equation [Eq. (13)]. The signs of these coordinates are determined by recourse to other data. The magnitude and sign of the coordinates of two atoms are, then, still undetermined. Substitution of the $n-2$ known values into the moment equations for the original molecule,

$$\sum_i m_i z_i = 0, \quad \text{and} \quad \sum_i m_i z_i^2 = I_Y,$$

yields two equations in the two unknown coordinates. These two equations give two sets of solutions for the coordinates of the remaining two atoms: z_1', z_2' and z_1'', z_2'', where, in general, $z_1' \neq z_1''$ and $z_2' \neq z_2''$. By recourse to other data it should be possible to choose the correct one of these two sets. If only $n-3$ substitutions were made, the moment equations of the original molecule would yield two equations in three unknowns. Solution would then be impossible. Hence, the minimum number of isotopic substitutions required to determine completely the molecular configuration of a linear molecule is $n-2$.

For a *planar asymmetric top* molecule, $n-2$ isotopic substitutions would allow the magnitude of the x and y coordinates of these $n-2$ substituted atoms to be determined from the parameter equation [Eq. (20)]. The signs of these coordinates would again be determined from other sources. The coordinates of the remaining two atoms can be determined by solving the five moment equations of the original molecule:

$$\sum_i m_i x_i = 0, \quad \sum_i m_i x_i^2 = I_Y, \quad \sum_i m_i y_i = 0,$$
$$\sum_i m_i y_i^2 = I_X, \quad \sum_i m_i x_i y_i = I_{xy} = 0.$$

These five equations again yield two possible sets of solutions: x_1', x_2', y_1', y_2' and x_1'', x_2'', y_1'', y_2''. By recourse to other data it should be possible to choose the correct one of the two sets. Consideration of the data obtained from $n-3$ substitutions again indicates that the available information is not sufficient to fix the six coordinates of the remaining three unsubstituted

[9] Townes, Holden, and Merritt, Phys. Rev. **74**, 1113 (1948).

[10] D. K. Coles and R. H. Hughes, Phys. Rev. **76**, 178 (1949).

atoms. Hence, as for linear molecules, at least $n-2$ substitutions are required to completely determine the structural configuration.

A similar analysis for a *nonplanar asymmetric top* molecule leads to quite different conclusions. In this case all nine moment equations in Eq. (10) are available—a somewhat simpler set can be obtained from the three in Eq. (10a) and the six in Eq. (21). If $n-3$ substitutions are made, the three Cartesian coordinates of each of the substituted atoms can be determined from parameter Eqs. (23) or (25). The signs of these coordinates would again be determined from other sources. Substitution of these data into the nine moment equations yields nine equations in nine unknowns—the unknowns being the coordinates of the remaining three unsubstituted atoms. Since six of these equations are of the second degree and three of the first degree, an exact solution of these equations would be quite difficult. It would probably be more expedient to use a numerical technique in solving these equations; the initial approximate values for such a process could be obtained from other sources of data. Hence, it should be possible to completely determine the structural configuration from only $n-3$ substitutions. Obviously if measurements from $n-2$ substitutions were available, the determination would become vastly simpler and the need for accurate data from other sources greatly reduced.

For all types of molecules, when information from $n-1$ substitutions is available, the determination becomes still simpler. In this case data from other sources are helpful aids but are not absolutely necessary. Molecules possessing certain elements of symmetry will require an even smaller number of substitutions than the minima indicated above. Finally, it should be mentioned that when effective moments of inertia are used and the calculations are first made using the minimum number of substitutions and then repeated using a larger number of substitutions, exact agreement should not be expected unless the effects of zero-point vibrations are taken into account.

IX. MASS DIFFERENCE RATIOS

If more than one isotopic substitution has been made on the same atom, the additional information can be used to determine mass difference ratios.[9] Thus, if I_X, I_Y, I_Z represent the moments of inertia of an original molecule of mass M, and I_X', I_Y', I_Z' and I_X'', I_Y'', I_Z'' represent the moments of inertia of isotopically substituted species of this molecule in which the mass of the ith atom has been changed by $\Delta m'$ and $\Delta m''$, respectively, then from Eq. (26),

$$\begin{aligned} r^2 &= (2\mu')^{-1}(I_X'-I_X+I_Y'-I_Y+I_Z'-I_Z), \\ &= (2\mu'')^{-1}(I_X''-I_X+I_Y''-I_Y+I_Z''-I_Z), \end{aligned}$$

where $\mu'=M\Delta m'/M+\Delta m'$ and $\mu''=M\Delta m''/M+\Delta m''$. Hence, the mass difference ratio is given by

$$\frac{\Delta m'}{\Delta m''}=\frac{M'(I_X'-I_X+I_Y'-I_Y+I_Z'-I_Z)}{M''(I_X''-I_X+I_Y''-I_Y+I_Z''-I_Z)}, \quad (27)$$

where $M'=M+\Delta m'$ and $M''=M+\Delta m''$. Analogous formulas can be obtained for linear, symmetric top, and planar asymmetric top molecules from Eqs. (13) and (19).

ACKNOWLEDGMENT

The author is very grateful to Dr. D. K. Coles for many helpful discussions.

In the modern theory there is no longer any attempt to make an imaginative picture of the atom. An atom only gives evidence of its existence when it emits energy, and therefore experimental evidence can only be of changes of energy. The new theory takes over from Bohr the doctrine that the energy in an atom must have one of a discrete series of values involving h; *each of these is called an "energy level."*—BERTRAND RUSSELL, Human Knowledge, 1948.

Particle Properties of Radiant Energy in Wave Guides

WALTER R. RAUDORF*
University of British Columbia, Vancouver, British Columbia, Canada

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The mean energy per excitation wavelength of e.m. radiation propagated along a lossless uniform wave guide is found to be $w=Q\nu$, Q being independent of frequency for a given amplitude of the longitudinal field intensity, mode of the field, and structure of the guide. The frequency ν and the group velocity v_g of the radiant energy are related by the equation $\nu/\nu_c=[1-(v_g/c)^2]^{-\frac{1}{2}}$, where $\nu_c=c/\lambda_c$ denotes the frequency at cutoff, and λ_c the excitation wavelength at cutoff. This equation permits one to put

$$h\nu=h\nu_c+h\nu_c\{[1-(v_g/c)^2]^{-\frac{1}{2}}-1\}.$$

The amplitudes of the field intensities are chosen such that Q equals Planck's constant h. Applying the mass-energy relation, the term $h\nu_c=hc/\lambda_c=m_0c^2$ may be interpreted as the "potential" (=rest) energy of the quantum $h\nu$ and m_0 as the rest mass. The momentum p associated with $h\nu$ is found to be $p=h/\lambda_g$, λ_g being the wavelength along the guide. When radiant energy passes from one guide into another one of different cross-sectional structure via a tapered piece of wave guide, a partial conversion of "potential" to "kinetic" energy, and vice versa, occurs along the tapered section. In the case of *TEM* waves, which, however, can only be supported in a guide of infinite transverse dimensions, the group velocity becomes c and the rest mass zero.

IN an interesting paper[1] on particle and quantum properties of electromagnetic wave fields in free space, H. W. König attempts a classical interpretation of the quantum nature of light which allows the possibility of a rest mass for the photon. His conclusions, however, are based on several arbitrary assumptions and do not appear to be consistent with results of nonclassical theory (see reference 1, page 420).

It is shown in the following that such particle properties of electromagnetic wave fields can be derived classically without any arbitrary assumptions and in accordance with established theories, if a geometrical structure, such as a wave guide, imposing periodicity requirements on the field, is present.

For the sake of simplicity, a uniform nondissipative wave guide of rectangular cross section may be considered. The perfectly conducting walls are the planes $x=0$ and $x=a$, $y=0$ and $y=b$ (Fig. 1). The z axis is the axis of propagation along which $H_{m,n}$ waves may be propagated. The field distribution is given by the following equations (MKS units):[2]

$$\left.\begin{aligned}
E_x&=A\omega\mu\frac{n\pi}{b}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin(\omega t-\beta z),\\
E_y&=-A\omega\mu\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin(\omega t-\beta z),\\
E_z&=0\\
H_x&=A\beta\frac{m\pi}{a}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin(\omega t-\beta z),\\
H_y&=A\beta\frac{n\pi}{b}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin(\omega t-\beta z),\\
H_z&=-Ak^2\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\cos(\omega t-\beta z).
\end{aligned}\right\}\quad(1)$$

In Eqs. (1) A, in amp-m, is an arbitrary constant determining the amplitude, and m and n are integers defining the mode of transmission by indicating the number of half-period variations in transverse field intensity along the x and y dimensions of the guide, respectively. The

* Now at Sir George Williams College, Montreal, Canada.

[1] H. W. König, Acta Physica Austriaca **4**, 405 (1951).

[2] L. G. Huxley, *The Principles and Practice of Wave Guides* (Cambridge University Press, Cambridge, 1947), p. 52.

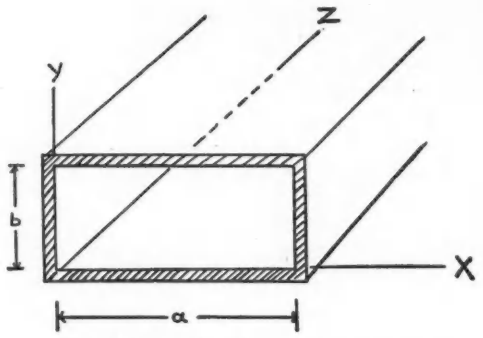


FIG. 1. Illustrating the notation used with rectangular guides.

other symbols used in these equations and in the further discussion are as follows: E, H are electric and magnetic field intensity, respectively, $\nu=\omega/2\pi$ frequency, ν_c the so-called "cut-off frequency" (see below), λ_c wavelength at cut-off frequency, λ wavelength of field excitation or free space wavelength, λ_g wavelength along the guide, v_p phase velocity, v_g group velocity, ϵ, μ dielectric constant and permeability in free space, respectively, and $c=(\epsilon\mu)^{-\frac{1}{2}}$ velocity of light in free space. The quantity k is given by $k^2=(m\pi/a)^2+(n\pi/b)^2=(2\pi/\lambda_c)^2$, and $\beta=2\pi/\lambda_g$. For any mode of transmission, the cut-off frequency is the lowest possible frequency at which such mode is possible; the cut-off wavelength λ_c is then given in terms of the guide dimensions a and b by

$$(1/\lambda_c)^2=(m/2a)^2+(n/2b)^2. \tag{2}$$

The quantities λ, λ_g, and λ_c are related by the equation

$$(1/\lambda_c)^2=(1/\lambda)^2-(1/\lambda_g)^2, \tag{3}$$

or, multiplied by $(1/\nu)^2$,

$$(1/\nu\lambda_c)^2=(1/c)^2-(1/v_p)^2, \tag{4}$$

where $v_p=\lambda_g\nu$ and $c=\lambda\nu$. Frequency and wavelength at cutoff are linked by the equation

$$c=\lambda_c\nu_c. \tag{5}$$

It is important to note that Eqs. (3), (4), and (5) hold quite generally for any kind of uniform guide and mode.[3] The choice of the cross-sectional structure (boundary conditions) and mode of field only affect the value of λ_c. As c and λ_c are independent of ν, it follows that the phase velocity v_p depends on frequency, and the propagation is accompanied by dispersion. The group velocity is defined by $1/v_g=(d/d\nu)(\nu/v_p)$, which yields, with Eq. (4),

$$v_pv_g=c^2. \tag{6}$$

The mean energy W per unit length in the guide is found in the usual way to be

$$W=A^2k^2ab\omega\mu\beta v_p/8c^2, \tag{7}$$

and the total average power flow P along the guide in the z direction is

$$P=A^2k^2ab\omega\mu\beta/8, \tag{8}$$

or, with Eqs. (6) and (7), $P=Wv_g$, which states that radiant energy in the guide is transmitted with group velocity. The mean energy w per excitation wavelength λ is given by

$$w=\lambda A^2k^2ab\omega\mu\beta v_p/8c^3,$$

or rearranged,

$$w=(A^2k^2ab\mu\pi^2/2c)\nu=Q\nu,$$

where

$$Q=A^2k^2ab\mu\pi^2/2c. \tag{9}$$

It is obvious that Q is a constant independent of ν for a given value of A, mode of field and geometry of guide, and that Q has the dimensions of Planck's constant h. The radiant energy w, therefore, may be expressed in terms of $h\nu$, giving

$$w=nh\nu, \tag{10}$$

n being defined by $n=Q/h$. As w represents the mean energy per excitation wavelength along the guide, the numeral n is not necessarily an integer.

Since a signal, i.e., finite amount of energy, cannot be transmitted with a strictly monochromatic wave, the equation $w=Q\nu$ may be regarded as the definition of a mean frequency of a group of waves, if in Q an appropriate average value of A is substituted. (Also, vice versa, using an appropriate average of ν, $w=Q\nu$ may be regarded as the definition of Q and, thence, by Eq. (9) of A). Instead of the total cross-sectional area of the guide, one could choose equally well the smaller area between two pairs of planes within the guide, which are

[3] N. Marcuwitz, *Wave Guide Handbook*, M.I.T. (McGraw-Hill Book Company, Inc., New York, 1951), p. 8.

parallel to the actual walls of the guide and along which the necessary boundary conditions are fulfilled, e.g., $x=a/m$ and $x=2a/m$, $y=b/n$ and $y=2b/n$. Apart from a reduction of the value of Q, the decisive equations would not be affected.

In order to avoid the factor n in the following equations, let the special case now be considered where the arbitrary constant A in the expression for Q has such value that Q becomes equal to Planck's constant h; hence $n=1$ and $w=h\nu$. The value of A is determined by the power output of the generator at the sending end of the guide. Since the output of the generator can be attentuated at will, the integration constant A may be regarded as arbitrary. It is obvious that a variation of frequency will not affect the value of A, if during such variation not the power output and not the amplitude of the electric or transverse magnetic field is kept constant, but rather the amplitude of the longitudinal magnetic field H_z.

According to Maxwell's theory, the z component of the electromagnetic momentum p_z is originally defined by the equation

$$c^2 p_z = \int_0^a \int_0^b \int_0^\lambda \bar{S}_z dx dy dz,$$

$\bar{S}_z$ being the mean z component of Poynting's vector. Employing Eqs. (8), (9), and (10) with $n=1$, it is readily found that

$$p_z = h/\lambda_g. \tag{11}$$

On the other hand, the z component of the momentum of the mass m which is associated with the quantum of energy $w=h\nu$ is equal to

$$p_z = m v_g, \tag{12}$$

since w, as an element of W, moves with group velocity v_g. Equalizing Eqs. (11) and (12) and substituting c^2/v_p for v_g, one obtains

$$m = h\nu/c^2. \tag{13}$$

Equations (11) and (13) serve to determine mass and momentum associated with the electromagnetic energy $h\nu$ streaming along the guide.

The next step consists in referring the wave field, represented by Eqs. (1), to a coordinate system $K_0(X, Y, Z_0)$ that moves with group velocity v_g in the direction of propagation relative to the coordinate system $K(X, Y, Z)$ of the guide. With respect to K_0 the group velocity of the wave field becomes zero and the phase velocity $(v_p)_0$, because of Eq. (6), infinitely large. The frequency ν_0 observed in K_0 is defined by Eq. (4). Putting $(v_p)_0=\infty$ and $c=\lambda_c\nu_c$, one obtains $\nu_0=\nu_c$, in accordance with the definition of the cut-off frequency. For the wavelength along the guide one gets $(\lambda_g)_0=\infty$, because of $(v_p)_0=\nu_0(\lambda_g)_0=\infty$. The mode of the wave field observed in K_0 is obtained by putting $\beta=0$ and $\omega=\omega_0=\omega_c$ in the field equations [Eqs. (1)]. Thus,

$$\left.\begin{aligned}
(E_x)_0 &= A\omega_c\mu \frac{n\pi}{b} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\omega_c t_0,\\
(E_y)_0 &= A\omega_c\mu \frac{m\pi}{a} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \sin\omega_c t_0,\\
(E_z)_0 &= 0,\\
(H_x)_0 &= (H_y)_0 = 0,\\
(H_z)_0 &= -Ak^2 \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \cos\omega_c t_0.
\end{aligned}\right\} \tag{14}$$

Obviously there is no phase dependence on the axial distance; hence, there is no progressive wave in K_0. A given field component oscillates everywhere in phase with itself. The components of E vibrate in quadrature with the z component of H. There is no power flow along the z_0 axis. This, incidentally, is the mode of field which De Broglie associated with a particle at rest.[4] It is worth noting that the Lorentz transformation applied to the field equations [Eqs. (1)] for the transition from K to K_0 leads exactly to Eqs. (14). The excitation wavelength observed in K_0 becomes λ_c, and the energy per excitation wavelength becomes $w_0=h\nu_c$.

Mass and momentum associated with w_0 appear to be, respectively,

$$\begin{aligned} m_0 &= (h\nu_c)/c^2 = h/(c\lambda_c),\\ p_0 &= 0. \end{aligned} \tag{15}$$

Using Eqs. (5) and (6) one readily finds from Eq. (4) that

$$\nu/\nu_c = [1-(v_g/c)^2]^{-\frac{1}{2}}. \tag{16}$$

[4] A. March, *Quantum Mechanics of Particles and Wave Fields* (Chapman and Hall, Ltd., London, 1951), p. 12.

Dividing Eq. (13) by Eq. (15) gives the relationship between mass in motion and mass at rest, $m=m_0[1-(v_g/c)^2]^{-\frac{1}{2}}$, a result which may be obtained also by subjecting energy and momentum to the Lorentz transformation.

Product $h\nu$ represents the total quantity of energy w in K, relative to which it moves with group velocity, whereas $h\nu_c$ represents the energy of this energy "quantum" observed in a reference system relative to which it is at rest. Therefore, $h\nu_c$ may be interpreted as the potential[5] energy w_p of this "quantum." The kinetic energy w_k appears to be

$$w_k \equiv w - w_p = h\nu - h\nu_c = h\nu_c[(\nu/\nu_c)-1],$$

or with Eqs. (16) and (15),

$$w_k = h\nu_c\{[1-(v_g/c)^2]^{-\frac{1}{2}}-1\} = m_0c^2\{[1-(v_g/c)^2]^{-\frac{1}{2}}-1\}.$$

Equation (15) states that the rest mass, and thus the potential energy m_0c^2, varies with the cut-off wavelength, which depends on the structure of the guide and the mode of the field. This is understandable, in view of the fact that the group velocity of the electromagnetic energy in a guide depends on λ_c, according to Eq. (4), which by Eq. (6) may be rewritten as

$$v_g = c[1-(\lambda/\lambda_c)^2]^{\frac{1}{2}}. \qquad (17)$$

When radiant energy passes from one guide into another one of larger cross-sectional area via a tapered piece of guide, so that the mode of transmission does not change, the group velocity of the radiation and hence its kinetic energy in the larger guide, is greater. An increase in kinetic energy, however, can only occur at the expense of potential energy, as the total energy must be preserved. The tapered guide section apparently acts as a transformer, converting potential energy of the radiation partially into kinetic energy and vice versa, according to the direction of propagation. An expansion of the electromagnetic field, however, reduces the field intensities. Therefore, the quantity $Q=\lambda_g P/\nu = \lambda^2 P/v_g$ in the larger guide will also be smaller, and hence $n<1$. This means that the section of the larger guide which contains the mean energy $w=h\nu$ will be longer by a factor $1/n$. In other words, the mean axial distance between two successive quanta is larger in the wider guide as a consequence of the acceleration occurring along the tapered section.

The dominant mode in free space, which may be regarded as a uniform wave guide of infinite transverse dimensions[3] is the "*TEM* mode" (transverse electromagnetic waves), which a hollow pipe wave guide is not capable of supporting. In this case the cut-off wavelength becomes infinitely large, and therefore,

$$v_p = v_g = c,\ \nu_c = 0 \text{ and } w_p = 0,\ m_0 = 0.$$

All the potential energy changes into kinetic energy as the radiation passes from confined space into free space. A quantum traveling at speed c has attained its maximum speed and possesses neither potential energy nor rest mass.

In conclusion it may be pointed out that the group velocity is almost equal to c, if $\lambda_c \gg \lambda$. Unless the cross-sectional dimensions of the guide are microscopical, it is immaterial whether the Compton effect, for example, is studied with or without the employment of wave guides.

[5] The term "potential energy" w_p is used here for what in the theory of relativity is conventionally called "rest energy" w_0; w_p is the maximum energy the quantum can possess when at rest.

Functional Writing

At Colgate University, English department consultants take an opaque projector into science laboratories and other classrooms and show on a screen samples of student reports written for these courses. The papers are discussed by the class and criticized both by the English and the subject-matter teachers. This is part of the Colgate plan of functional writing, in which emphasis is placed on accurate and concise expression in all courses. "If it isn't good English it isn't good physics either" is the slogan.

On Fluid Dynamics in Physics Teaching*

R. J. SEEGER†
Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland
(Received June 4, 1952)

In the light of private answers suggested for the public questions: What is physics? What is education?, a critical review is made of fluid dynamics in physics teaching. First, general physics is examined for what is being done in teaching fluid dynamics in physics. A critical analysis is made in particular of the important Bernoulli theorem as to what it is, what it says, what it means, and how it is used. Second, theoretical physics is examined for what is being done. In this case, the motion of a sphere has been selected as illustrative of present teaching. The typical course in theoretical physics is then examined with respect to its material about the motion of a sphere, both in a perfectly inviscid fluid and in a perfectly viscous fluid. This academic material is compared finally with experimental observations, particularly those in wind tunnels and in ballistics ranges. In conclusion, questions are noted on the foundations of fluid dynamics, particularly the role of viscosity, turbulence, the concept of a steady state, and the role of heat. The review ends with a moral question as to the students' stake.

PHYSICS AND EDUCATION

THE subject of this discussion revolves about two fundamental questions. What is physics? What is education? In general, these questions lead to a third: how to teach physics, or more specifically, how to teach fluid dynamics in physics? In this connection, there is an interesting comment by F. K. Richtmyer[1] in the first article in the *American Physics Teacher*, entitled "Physics is Physics," with reference to a remark by G. W. Jones, Professor of Mathematics at Cornell University: "to become a successful teacher of mathematics one must acquire a thorough knowledge of mathematics—and then by constant practice acquire the art of teaching it." (The latter part of this statement is due to Richtmyer.) He states further "teaching is an art and not a science." On the other hand, H. G. Wells has emphasized, "No man can be a good teacher when his subject becomes inexplicable." Here we are faced with the dual implication of our question, namely, that science signifies essentially knowledge derived from nature, and art in this case is the method of its expression.

To the above public questions every teacher has rightfully his own private answers, which form the axiomatic foundation for all his instruction. Accordingly, I must state at once my own views as to the primary goal of physics teaching and the methods of its attainment.

Someone once asked the artist H. Matisse how he saw an orange. "That depends," he answered, "on whether I am buying it, eating it, or painting it." Likewise, our outlook on physics depends upon our point of view. Evidently any matter, including physics, can be regarded from the viewpoints of aesthetic enjoyment, of rational inquiry, and of technological use. Any integrated education must involve all these aspects even though it may stress only one. Thus, A. N. Whitehead in his *The Aims of Education* says, "There is not one course of study which merely gives general knowledge and another which gives specific knowledge."[2] Even so-called pure science and applied science are intimately interrelated, as F. Klein emphasized long ago. W. F. G. Swann in his recent article "Teaching of Physics"[3] observes further that, "Physics has combined two phases, knowledge of facts and knowledge of ideas." He proceeds to emphasize the ideas aspect. G. F. B. Riemann, too, has stressed the latter in the statement, "Science is the attempt to comprehend nature by means of *concepts*." In the same spirit physics, to me, is primarily the *understanding of physical phenomena through concepts*.

* Address given at a symposium sponsored jointly by the American Association of Physics Teachers, the Fluid Dynamics Division of the American Physical Society, and the Physics Division of the American Society for Engineering Education, at Michigan State College, June, 1951.

† At present with the National Science Foundation, Washington 25, D. C.

[1] F. K. Richtmyer, Am. Phys. Teacher **1**, 1 (1933).

[2] A. N. Whitehead, *The Aims of Education* (Macmillan Company, New York, 1929).

[3] W. F. G. Swann, Am. J. Phys. **19**, 182 (1951).

TABLE I. Distribution of fluid dynamics topics in 46 general physics textbooks.

Topics	Number of texts
Bernoulli theorem	42
Applications	33
Torricelli theorem	32
Vena contracta	11
Viscosity	23
Friction head	18
Streamlining	14
Hydraulic ram	12
Poiseuille law	9
Turbulence	8
Water wheel	7
Venturi meter	6
Stokes law	3
Vortex motion	2
Reynolds number	2

Whitehead suggests further a method of attaining such a goal. He mentions "two educational commandments: do not teach too many subjects, and again, what you teach, teach thoroughly."[2] Thus, our first imperative is to select the elemental concepts and the associated phenomena. Physics, therefore, must be regarded not as a dogmatic presentation of a self-sufficient body of learning, but rather as a creative search for a predictive description of nature. It should strive continually for liberation from superstitions, both old and new. Thus, physics teaching of fluid dynamics should stimulate students to think critically about its problems. Breadth, moreover, is to be achieved through coherent ties rather than through widespread spottiness, as *e pluribus unum* should signify an orchestration rather than a melting pot. Above all, may I suggest that quality in teaching, as in acoustics, depends upon overtones and not just upon fundamentals.

Education, in short, is not so much like a static structure, which is fixed according to a blue print; it is more like an organic plant, which grows up and down and spreads out. From this point of view let us now regard fluid dynamics teaching at two levels: that of undergraduate general physics and that of graduate theoretical physics.

FLUID DYNAMICS IN GENERAL PHYSICS

First of all, let us review what is done in general physics courses by glancing at some elementary textbooks that have appeared between 1930 and 1951. It is significant that in fifteen textbooks (522 pages average) there is absolutely no mention at all of any topics in fluid dynamics, although the textbooks themselves have inviting titles like *Everyday Physics; Man and his World; The Universe; Atoms, Rocks, and Galaxies; From Galileo to Cosmic Rays*, etc. In my personal opinion the quality itself in some of these cases is poor because of the very overtones that are present or absent. For example, in one textbook there is a mention of a so-called Aristotelian law of falling bodies, but no word at all about Archimedes principle. Such a selection, I believe, gives a perverted idea as to the status of Greek science. In another textbook there is a description of the measurement of atmospheric pressure but no indication at all as to any physical meaning of the term pressure—not even a guiding definition.

In those 46 textbooks (634 pages average) which devote some part to fluid dynamics (including viscosity) there is an average of nine pages (1.42 percent) used for this subject. The impression gained from this cursory review of college physics textbooks is reflected in the *American Journal of Physics*, the organ of physics teachers. In its entire history since 1933 there have been only 27 articles and 21 letters (152 pages out of 6955 pages total, approximately 2.19 percent) that can be construed in any sense to be concerned with fluid dynamics. (As a matter of fact, three of the articles (57 pages) are actually on rockets.) Evidently, this *Journal*, as well as the textbooks, reflects merely the interest of physics teachers themselves, as emphasized also in the popular histories of physics. For example, F. Cajori[4] fails to mention the fluid dynamics contributions of L. Euler and D. Bernoulli in the 18th century, and of O. Reynolds, M. H. Navier, and G. G. Stokes in the 19th century. As a matter of fact, there is no reference to turbulence including Reynolds' investigations either in this book or in his complementary history of mathematics.[5] All in all, fluid dynamics has not been particularly interesting to physicists themselves.

[4] F. Cajori, *A History of Physics* (Macmillan Company, New York, 1929), rev. edition.

[5] F. Cajori, *A History of Mathematics* (Macmillan Company, New York, 1919), second edition.

Let us now turn to the topics of fluid dynamics that are actually selected in general physics textbooks. The findings are listed in Table I.

It is not surprising to find that flows in pipes and in channels, which are everyday experiences, are among the popular topics. The initial scientific achievement along this line was the series of investigations of J. L. M. Poiseuille (see simultaneous work of G. Hagen) begun in 1835, announced in 1840, and published in 1846.[6] Specifically, Poiseuille observed that the quantity of fluid flowing from a tube is directly proportional to the excess pressure across its ends and the fourth power of its diameter, and inversely proportional to its length. It is interesting that the motivation for this work was concerned originally with the problem of blood circulation in capillaries, but that later studies[7] show Poiseuille's results not to be valid for fine capillaries (about a 3-mm diameter is the lower limit for blood flow).

It is difficult to understand completely the pedagogical reason for the inclusion of the Poiseuille law in an elementary textbook. Perhaps the simplified theoretical deduction that can be made of the law is the primary objective. Surprisingly enough, few authors note that whereas Stokes undoubtedly derived the law (to date, one of the few existing solutions of the Navier-Stokes equations with boundary conditions), he himself never published his proof, apparently because of its disagreement with the available experimental data (primarily for large pipes). Accordingly, it is the later published derivation of G. Wiedemann and of E. Hagenbach[8] that is the one commonly used in textbooks today. What is truly disappointing, however, in an elementary textbook is the failure to stress—or even to note—that Poiseuille's original work is a classical example of the observational phase of the inductive method. In unparalleled fashion he isolated the important physical variables and analyzed the sources of error. His resulting empirical law for water flowing through fine capillaries (0.01 to 0.65 mm) had an accuracy between 0.1 and 0.2 percent. Students should be encouraged to read for themselves a translation of the original article of Poiseuille[9]—not to mention the equally stimulating presentations of B. Pascal.[10]

As early as 1854 and 1869 Poiseuille pointed out also that there are definite limitations to the validity of his formula. The classical work in this direction, of course, is that of O. Reynolds,[11] who began experimenting in 1880 with pipes $\frac{1}{4}$ in., $\frac{1}{2}$ in., and 1 in. in diameter and 4 ft 6 in. long. The work of Reynolds is concerned primarily with the limitation due to the initiation of turbulence (see Fig. 1). It is now known that Poiseuille's law for laminar flow is strictly valid only up to a certain "critical" Reynolds' number R_c of about 1000, although for smooth rounded pipes with special entrances the law may be valid up to a Reynolds number of 75 000. Two other physical factors may be important also. If, for instance, there is slipping along the walls, the quantity of fluid is classically proportional to the factor[12] $(1+4\mu/\lambda R)$, where μ is the coefficient of viscosity, λ a slip coefficient, and R the Reynolds number. Furthermore, as the pressure of a gas decreases so that molecular structure becomes significant, one has to multiply by the so-called Knudsen[13] factor (ratio of molecular mean free path to a body length, such as the diameter of the tube). All the above remarks

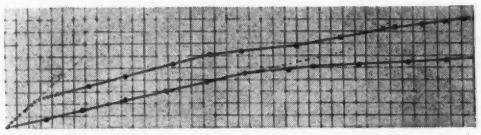


FIG. 1. Reynolds' data for the nonlinear variation of the quantity of liquid issuing from a tube with the speed (see reference 11).

[6] J. L. M. Poiseuille, Mémoires des Savants Étrangers 9, 433 (1846). G. Hagen, Pogg. Ann. Phys. Chem. 46, 423 (1839).

[7] G. Barr, *Viscometry* (Oxford University Press, New York and London, 1931).

[8] G. Wiedemann, Pogg. Ann. Phys. Chem. 99, 217 (1856). E. Hagenbach, Pogg. Ann. Phys. Chem. 109, 385 (1860).

[9] Rheological Memoirs: J. L. M. Poiseuille, *Experimental Investigations upon the Flow of Liquids in Tubes of Very Small Diameters*, translation by W. H. Herschel (Easton, Pennsylvania 1, V940).

[10] *The Physical Treatises of Pascal*, translation by I. H. B. and A. G. H. Spiers (Columbia University Press, New York, 1937).

[11] O. Reynolds, Trans. Roy. Soc. (London) 174, 935 (1883); Papers on Mechanics and Physics Subjects II, 51 (Cambridge, 1900–1903).

[12] H. A. Lorentz, *Lectures on Theoretical Physics* (Macmillan Company, New York, 1927), Vol. 1, p. 81.

[13] M. Knudsen, Ann. Physik 28, 75 (1909); *The Kinetic Theory of Gases* (Methuen and Company, London, 1946), second edition.

point up to a severe criticism of so many modern textbooks in that experimental information is rarely given completely or even referenced by a caution sign. In the case of the Poiseuille law, moreover, the historical significance of some of the great investigations in physics has also been neglected.

Even more popular than the Poiseuille law, it seems, is the *vena contracta*. Perhaps, the appeal here is the gobbledygook one of "far away places with strange sounding names." One is reminded of H. C. Andersen's story of the "Nightingale": "The music master wrote a twenty-five volume about the artificial bird. It was learned, long-winded, and full of hard Chinese words. Yet everyone said they had read and understood it, lest they should show themselves stupid." Sometimes the phrase *vena contracta* is used just as a label, occasionally it is defined, but hardly ever is one told how the concept originated or why it is now important.

The practical Romans, for example, were particularly concerned with the *vena contracta* in connection with their problem of increasing the rate of flow from orifices connected to their water supply. The inclusion of such a concept in an elementary course illustrates Whitehead's "inert ideas": "Above all things, we must beware of what I will call 'inert ideas'—that is to say, ideas that are merely received into the mind without being utilized, or tested, or thrown into fresh combinations."[2]

Finally, the selection of phenomena and of ideas for the fluid dynamics part of a general physics course is not obvious and certainly not modern. At best, it may be said to be classical. First class in education, however, may not necessarily mean the highest.

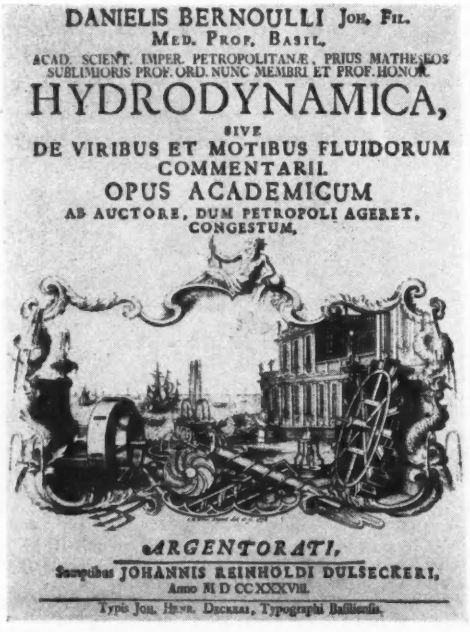

DANIELIS BERNOULLI Joh. Fil.
Med. Prof. Basil.
ACAD. SCIENT. IMPER. PETROPOLITANÆ, PRIUS MATHESEOS SUBLIMIORIS PROF. ORD. NUNC MEMBRI ET PROF. HONOR.
HYDRODYNAMICA,
SIVE
DE VIRIBUS ET MOTIBUS FLUIDORUM COMMENTARII.
OPUS ACADEMICUM
AB AUCTORE, DUM PETROPOLI AGERET, CONGESTUM.
ARGENTORATI,
Sumptibus JOHANNIS REINHOLDI DULSECKERI,
Anno M D CC XXXVIII.
Typis Joh. Henr. Decker, Typographi Basiliensis.

FIG. 2. Title page of Bernoulli's *Hydrodynamica* (see reference 17).

BERNOULLI'S THEOREM IN GENERAL PHYSICS

In view of the widespread consideration given to Bernoulli's so-called theorem, as well as its basic engineering importance, we shall consider this topic in some detail. A. Sommerfeld[14] insists: "Bernoulli's equation is the most important theorem in elementary fluid dynamics," but F. Magie does not bother even to mention it in his source book in physics.[15] An educational argument for its inclusion in an elementary course is that it is one of the few simple concepts concerned with the mechanics of a continuum as distinct from that of physical bodies. In this respect, it is strange that J. C. Maxwell in his *Matter and Motion* (1877) does not discuss fluids at all.

The first question that arises is whether the Bernoulli statement itself is essentially a theorem, a principle, or a law. First of all, what is a theorem? Is it "a general statement that has been proved"[16] particularly with respect to mathematics? On this basis it would appear historically, at least, that the relation is a theorem; for D. Bernoulli in his classical work[17] on hydrodynamics (see Fig. 2) utilized the principle of *vis viva*, first employed by Huygens, in order to show that the change in kinetic energy for a fluid element displaced during a constant discharge is equal to the work done on the element.

[14] A. Sommerfeld, *Mechanics of Deformable Bodies* (Lectures on Theoretical Physics) (Academic Press, Inc., 1950), Vol. 2.
[15] W. F. Magie, *A Source Book in Physics* (McGraw-Hill Book Company, Inc., New York, 1935).
[16] *Webster's New International Dictionary* (G. & C. Merriam Company, Springfield, 1949), second edition.
[17] D. Bernoulli, *Hydrodynamica* (Argentorati, 1738).

Sit velocitas aquæ in tubo E*d*, quæ nunc ut variabilis conſideranda eſt, $= v$: amplitudo tubi ut antea $= n$, longitudo E*c* $= c$: indicetur longitudo *ac* particulæ aqueæ infinite parvæ & effluxui proxime per dx: Erit guttula æqualis in E tubum ingreſſura eodem temporis puncto quo altera *acdb* ejicitur: dum autem guttula in E, cujus maſſa $= ndx$, tubum ingreditur acquirit velocitatem v, atque *vim vivam* $nvvdx$, quæ *vis viva* tota fuit de novo generata; nullum enim, ob amplitudinem vaſis A E infinitam, motum guttula in E habuit tubum nondum ingreſſa: huic *vi vivæ* $nvvdx$ addendum eſt incrementum *vis vivæ*, quod aqua in E*b* accipit, dum guttula *ad* effluit, nempe $2ncvdv$: aggregatum debetur *deſcenſui actuali* guttulæ ndx per altitudinem B E ſeu a: habetur igitur $nvvdx + 2ncvdv = nadx$ ſive $\frac{vdv}{dx} = \frac{a-vv}{2c}$.

In omni autem motu eſt incrementum velocitatis dv proportionale preſſioni ductæ in tempuſculum quod hic eſt $\frac{dx}{v}$: igitur in noſtro caſu eſt preſſio, quam guttula *ad* patitur, proportionalis quantitati $\frac{vdv}{dx}$, id eſt, quantitati $\frac{a-vv}{2c}$.

FIG. 3a. Text from Bernoulli's *Hydrodynamica* (see reference 17).

Inasmuch as Bernoulli used this *vis viva* principle throughout his entire book, it is difficult to locate the specific place where the so-called Bernoulli theorem can be said to be first introduced. My colleague, G. Kuerti, believes that page 258[17] of the *Hydrodynamica* is a likely location (see Fig. 3).

Secondly, what is a principle? Is it "a comprehensive law from which others are derived, or upon which others are founded"?[16] On this basis the mere fact that many textbooks derive the equally famous Torricelli theorem from the Bernoulli theorem would justify calling the latter a principle. Historically, however, this interpretation is not correct. E. Torricelli[18] himself obtained his law directly from observations of liquids flowing from orifices (here the concept of *vena contracta* is absolutely essential for the correct statement of the theorem). Indeed, the historical development of fluid dynamics concepts is from Torricelli through Bernoulli to Euler, whereas we usually trace the logical development from Euler through Bernoulli to Torricelli. It is for this reason that authors are at variance even as to the so-called "Father of Hydrodynamics." Thus, E. Mach[19] so desig-

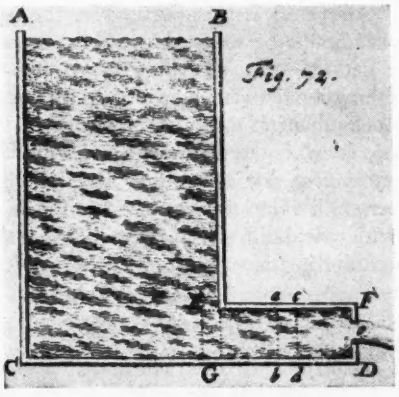


FIG. 3b. Diagram from Bernoulli's *Hydrodynamica* (see reference 17).

[18] E. Torricelli, *Lezioni Accademiche* (*Opere di Evangelista Torricelli*) (Faenza, 1919), Vol. 2, p. 185.

[19] E. Mach, *The Science of Mechanics* (Open Court, La Salle, 1942), fifth edition.

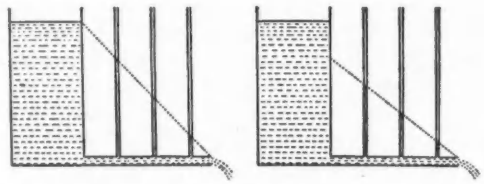


FIG. 4. Farwell's "Two diagrams appearing in different current textbooks" (see reference 22).

nates Torricelli, whereas E. T. Bell[20] points out that Bernoulli is often called the "Father of Mathematical Physics"; L. Prandtl and O. G. Tietjens[21] choose Euler.

Finally, what is a scientific law? Is it "the observed regularity in nature"?[16] If so, the Bernoulli relation is wanting in precise confirmation owing to the existence of frictional effects, which are usually ignored in the theoretical treatment of an idealized fluid. Incidentally, even when external friction is actually emphasized in an elementary course, another difficulty often arises from failure to specify the magnitude of the velocity head (see Fig. 4).[22]

The confusion indicated above illustrates, I believe, P. Frank's criticism of present-day teaching in his discussion of the "Philosophy of Science in the Physics Curriculum":[23] "The result of conventional science teaching has not been a critically minded type of scientist, but just the opposite···main source is the failure to teach students a clear distinction between the observation of facts and the drawing of logical conclusions." Above all, it is important for the teacher to discriminate between the roles of induction and of deduction in scientific research.[24] A valuable educational overtone is their interplay in the development of a useful theory.

There is an apparent uncertainty in an introductory course not only as to how to classify the Bernoulli relation, but even as to how to state it. Typical differences of opinion are shown by the following excerpts from three well-known texts:

Bernoulli's theorem, which applies only to an incompressible and nonviscous fluid.

Because liquids have so little compressibility, Bernoulli's principle holds more precisely for them than for gases.

The incompressible fluid Bernoulli equation can be interpreted as an energy equation, but the compressible fluid equation is only a momentum equation.

The point at issue here is not merely the semantics of a label, but actually the physics of its content. There is certainly need for reviewing elementary concepts of physics, too, from an advanced point of view.[25] In this connection, let us consider various pressure-speed relations pertinent to common fluid flows. For simplicity we shall confine our attention to the one-dimensional case.

We start with the basic dynamic relations for changes in mass (continuity equation), in momentum (force equation), and in energy (conservation of energy). One must distinguish between the equations of motion for two different ideal fluids, namely, for the perfectly inviscid incompressible fluid of L. Euler[26] and for the perfectly viscous incompressible fluid of C. L. M. H. Navier and of G. G. Stokes.[27] In both cases the equations can be readily modified for compressible fluids. In general, inclusion of viscous effects (for example, the coefficient of viscosity) necessitates consideration also of thermal conduction, inasmuch as the so-called Prandtl number (relating viscous and thermal conductive effects) is usually of the order of unity. Only if thermal conduction is negligible, moreover, does the energy equation reduce to the mechanical energy equation, which originates from a space integral of the force equation, provided all forces are included. For example, the time-rate of change of the mechanical energy (T kinetic, V potential) of a given quantity of a perfectly inviscid com-

[20] E. T. Bell, *Men of Mathematics* (Simon & Schuster, New York, 1937).
[21] L. Prandtl and O. G. Tietjens, *Fundamentals of Hydro- and Aerodynamics* (McGraw-Hill Book Company, Inc., New York, 1934).
[22] H. W. Farwell, Am. J. Phys. 12, 307 (1944).
[23] P. Frank, Am. J. Phys. 15, 202 (1947).
[24] R. von Mises, *Positivism* (Harvard University Press, Cambridge, 1952).
[25] F. Klein, *Elementary Mathematics from an Advanced Point of View*, translation by E. R. Hedrick and C. A. Noble. *Arithmetic, Algebra, Analysis* (Macmillan Company, New York, 1932); *Geometry* (Macmillan Company, New York, 1939).
[26] L. Euler, Acad. Berlin (1755). J. L. de LaGrange, Mém. Acad., Berlin (1781); *Mécanique Analytique* (Paris, 1788).
[27] C. L. M. H. Navier, Memoire de l'Académie Royal des Sciences 6, 389 (1823). G. G. Stokes, Trans. Cambridge Phil. Soc. 8, 287 (1845).

pressible fluid is given by

$$d/dt(T+V)=\int_s p\mathbf{q}\cdot\mathbf{d}\sigma+\int_v p\nabla\cdot\mathbf{q}d\tau,$$

where p is the pressure and $\mathbf{q}$ the fluid velocity (vector) in a region of volume v and of external surfaces s. The energy equation can be modified to include thermodynamic changes.

Pressure-speed relations along a streamline for one-dimensional steady (no time variation at a fixed point in space) motion are given below for different kinds of fluids at a height z above the level of zero potential energy:

(*a*) *Inviscid, incompressible fluid,*

$$\tfrac{1}{2}q^2+gz+\frac{p}{\rho}=A.$$

(*b*) *Inviscid, compressible fluid,*

$$\tfrac{1}{2}q^2+gz+\int dp/\rho=B,$$

where the pressure is a function of the density only (so-called polytropic changes). If $p=k\rho^\gamma$, then

$$\tfrac{1}{2}q^2+gz+\frac{\gamma}{\gamma-1}\frac{p}{\rho}=B.$$

For a perfect gas with negligible viscosity, γ is the adiabatic exponent (ratio of the specific heat at constant pressure to the specific heat at constant volume).

(*c*) *Viscous, incompressible fluid,*

$$\tfrac{1}{2}q^2+gz+p/\rho=A+\nu F,$$

where ν is the kinematic coefficient of viscosity (ratio of coefficient of viscosity to density) and F is a function of both the fluid rotation and the streamline curvature.

The parameter A (and B) is constant only along a particular streamline. Its gradient, however, can be shown to be equal to the sum of a term involving rotation and of another term involving the entropy. Thus, for irrotational, isentropic flow the value of such a parameter is uniform for all streamlines (this fact is sometimes called the strong form of Bernoulli's theorem).

For unsteady irrotational motion of an inviscid compressible fluid the following relation holds:

$$\tfrac{1}{2}q^2+gz+\int\frac{dp}{\rho}=C+\frac{\partial\phi}{\partial t},$$

where the value of C is uniform throughout the fluid and where ϕ is the velocity potential.

The uncertainty about the Bernoulli theorem in a general physics course extends not only to its mathematical formulation, but also to its physical interpretation. It revolves largely about the phrase "pressure energy," which is sometimes used to designate the term involving the pressure. (Three of the 21 letters in the *American Journal of Physics* deal with this subject.) The most famous public argument along this line took place in 1925 between E. H. Kennard and W. S. Franklin.[28] The former argues that the pressure term is certainly not the potential energy of compression and cannot strictly be said to belong to the moving fluid; rather it is "transmitted through the fluid." Franklin, however, insisted that potential energy is never localized in space, but "an idea which makes up for things ignored" (see H. Reichenbach's[29] introduction of the concept of interphenomenon to provide for just such abstractions as localization of potential energy). In this connection one is reminded of E. J. Routh (1876), who pointed out that so-called ignorable or cyclic dynamical coordinates result in an expression for the kinetic energy of spin that is formally equivalent to an additional potential energy. On this basis J. J. Thomson (1886) proposed to explain all potential energy by concealed motions, thus stamping an ignoramus on any hope of locating it. It would seem that the usage of the term pressure energy enjoys a certain slang value, but that it is physically confusing owing to the actual doing of external work by the moving fluid particle. (Pressure actually plays the mathematical role of a Lagrangian multiplier in the derivation of Euler's equations of motion from Hamilton's principle.) Indeed, from a purist point of view, pressure itself is a misleading name in connection with energy inasmuch as it is only a change in stress which is associated with work.

[28] E. H. Kennard, Science 62, 243 (1925); W. S. Franklin, Science 62, 397 (1925).

[29] H. Reichenbach, *Philosophic Foundations of Quantum Mechanics* (California, 1946).

I personally cannot leave this topic of Bernoulli's theorem without urging some modernization in the use of illustrative applications of it. In an age of supersonic flight why not call attention to certain important features of supersonic flow inherent in the following form of Bernoulli's theorem for air (negligible viscous effects) flowing at a constant gravitational level:

$$\tfrac{1}{2}q^2+\frac{7p}{2\rho}=B, \quad \gamma=1.4.$$

First of all, we note that whereas there is no limit to the value of the Mach number (the ratio of the flow speed to the acoustic speed; its square is proportional to the ratio of the fluid's available kinetic energy to its random thermal energy), there is an absolute limit to the flow speed of a fluid initially at rest (about 751 m/sec for air originally at room temperature). (An amusing paradox in the use of a concept like Mach number is that the effective Mach number of a projectile, which would have a Mach number of two if fired under room conditions, has a Mach number that is neither 0 nor four when fired into a supersonic stream flowing with a Mach number two.)

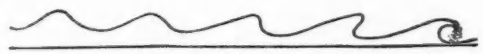


FIG. 5. Maxwell's picture of Rankine's conception of the formation of a shock wave (see reference 32).

Secondly, a flow speed equal to the acoustic speed is in a real sense a "critical speed," inasmuch as flows with speeds higher than this speed are supersonic and behave quite differently from subsonic flows. For example, if one considers the equation of continuity in the hydraulic form ρqs is constant (s is the cross-sectional area), one can show that for a convergent nozzle a supersonic flow decreases monotonously in speed but a subsonic flow increases; on the other hand, for a divergent nozzle a supersonic flow increases but a subsonic flow decreases.

Supersonic flow, moreover, is associated with shock waves, which being waves of finite amplitude are an actual occurrence rather than the idealized (mathematical) acoustic waves of infinitesimal amplitude. Although the basic equations[30] describing shock-wave phenomena were questioned by so eminent an authority as H. Lamb[31] as late as 1932, their physical formation was understood by W. J. M. Rankine as early as 1869. Figure 5[32] shows how the greater speed of the more displaced fluid results in a steeper wave front. More recently, the study of shock-wave interactions[33] has revealed new physical phenomena. For example, shock waves not only differ quantitatively with respect to the acoustic laws of regular refraction and reflection [Fig. 6a], but actually exhibit qualitatively distinctive patterns such as Mach intersections [Fig. 6b]. It is believed that such modern illustrations, supplemented with experiments in a shock tube[34] (the physicists' wind tunnel), will definitely serve to make fluid dynamics a "live" subject in physics.

[30] H. Hugoniot, J. École Polyt. **57–59** (1887–89).

FLUID DYNAMICS IN THEORETICAL PHYSICS

Let us now turn our attention to the treatment of fluid dynamics in a typical first-year graduate course such as the one commonly called "Theoretical Physics." First of all, what is being done?

Most of the fluid dynamics of such a graduate course is customarily concerned with a perfectly inviscid incompressible fluid. One starts with the basic equations of continuity and of motion. Laplace's partial differential equation follows from these if the fluid is assumed to be incompressible and irrotational. The major portion of this part of an American course is then devoted to the elegant mathematics associated with two-dimensional flows satisfying Laplace's equation, namely, the velocity potential and the stream function, conjugate functions, conformal mapping, the method of images, etc. Kelvin's circulation[35] and Helmholtz's vortices[36] form a natural mathematical continuation. Applications are usually restricted to the motion of a

[31] H. Lamb, *Hydrodynamics* (Cambridge University Press, Cambridge, 1932), sixth edition.

[32] J. C. Maxwell, *Theory of Heat* (Longmans, Green and Company, London, 1871).

[33] H. Polachek and R. J. Seeger, Proc. Symposia in Applied Math. (American Mathematical Society, 1949), Vol. 1, p. 119; J. Appl. Phys. **22**, 640 (1951); Phys. Rev. **84**, 922 (1951).

[34] W. Bleakney and A. H. Taub, Revs. Modern Phys. **21**, 584 (1949). Bleakney, Weimer, and Fletcher, Rev. Sci. Instr. **28**, 807 (1949).

[35] W. Thomson (Kelvin), Trans. Roy. Soc. (Edinburgh) **25** (1869).

[36] H. von Helmholtz, Crelle's Journal **55**, 25 (1858).

FIG. 6a. White-light interferogram of regular shock-wave intersection—Mach number 2.48, Reynolds number 10^6 (NOL Aerophysics Tunnel No. 3).

sphere, with particular attention to the virtual mass associated with unsteady conditions. Occasionally, there is a discussion of wave motion and of tides. It is quite disconcerting, however, to find little reference to Helmholtz's discontinuous motions[37] and to modern nonlinear problems, involving the possibility of specific principles of superposition (to replace the general principle of linear superposition first used by D. Bernoulli). Gas dynamics, dealing with compressible fluids, is usually ignored.

In the case of a perfectly viscous incompressible fluid one usually derives the Navier-Stokes equations of motion and then uses them to deduce the Poiseuille law for a tube and the Stokes law for a sphere. In neither application, however, is there usually any discussion of the important phenomenon of turbulence or of the significant role of the Reynolds number. Inhomogeneous fluids[38] (non-Newtonian) such as sols and gels are not even mentioned.

It is not surprising that such superficial regard for phenomena *per se* has evoked severe criticism as exemplified by the hyperbolic expressions below. E. T. Bell in his sketch of L. Euler under the heading "Analysis Incarnate" says: "The physical universe was an occasion for mathematics to Euler, scarcely a thing of much interest in itself; and if the universe failed to fit his analysis, it was the universe which was in error."[20] H. Rouse states: "During the past 150 years mathematicians and mathematical physicists have attempted completely rational solutions of the problems of fluid motion without any recourse to experimental measurements. These rational methods were made possible, however, only by such simplifying assumptions that the resulting solutions often bore little or no resemblance to the phenomena in question."[39] R. von Mises claims in 1945: "There exists so far no mathematical theory that would account satisfactorily for the actually observed types of fluid motion."[40] L. Prandtl and O. Tietjens remark: "A great amount of literature exists on the theoretical solution around moving bodies in the ideal fluid. However, we shall not dwell on these theories here, since they do not lead to a solution which conforms in the least with actually occurring flow

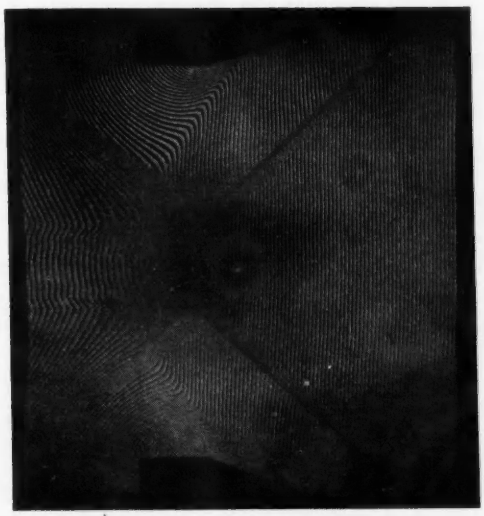


FIG. 6b. Monochromatic interferogram of Mach shock-wave intersection—Mach number 2.48, Reynolds number 10^6 (NOL Aerophysics Tunnel No. 3).

[37] H. von Helmholtz, Ber. Akad. Wiss., Berlin 215 (1868).

[38] E. G. Richardson, *Dynamics of Real Fluids* (Edward Arnold and Company, New York, 1950).

[39] H. Rouse, *Fluid Mechanics for Hydraulic Engineers* (McGraw-Hill Book Company, Inc., New York, 1938).

[40] R. von Mises, *Theory of Flight* (McGraw-Hill Book Company, Inc., New York, 1945).

phenomena."[41] S. Goldstein, too, concludes: "In this book we are concerned with the motion of fluids whose viscosity is small; the phenomena in such motions sometimes approximately agree, sometimes violently disagree with the predictions of ideal fluid theory.···In these circumstances it is surprising not that the results of theory and observation of disagree violently, but rather that they agree fairly well, as for the motion of airship bodies."[42] (Experimental observations for fluids with small viscosity are well described by theory up to a Mach number of approximately 0.25 except for boundary layer and wake phenomena.)

Without apologizing for the above criticisms we shall review for ourselves briefly the theoretical motion of a particular body in a perfectly inviscid fluid and in a perfectly viscous fluid. We choose a sphere for this purpose because its symmetry aids in laboratory control and calibration and because its simplicity affords little opportunity for misunderstanding. The sphere, moreover, is of specific interest alike to mathematician, physicist, aerodynamicist, meteorologist, and astronomer. In each case we shall look also at some observational data. Finally, we shall mention some well-known questions pertinent to the foundations of fluid dynamics.

SPHERE IN A PERFECTLY INVISCID FLUID

A practical problem of everyday concern is drag. Let us consider first a sphere moving at constant speed in a perfectly inviscid fluid. Note that the motion of the fluid with respect to the body is strictly steady motion, but that the motion of the fluid with respect to a fixed point infinitely distant (body moving) is not steady. (This distinction may be of considerable significance for turbulent phenomena.)

First of all, let us recall that, in general, there are three possible irrotational motions:

(1) For simple potential (Dirichlet) motion of an inviscid, incompressible fluid (probably theoretically true also for a compressible fluid), without acceleration and without free or solid surfaces, there is no force (but possibly a couple, if the body is not a sphere). This so-called paradox of J. LeR. d'Alembert (or P. L. Dirichlet) is evident for a sphere in that no change in momentum exists across an enveloping surface at a great distance away, inasmuch as the fluid motion here can be regarded as the superposition of a uniform motion and that of a dipole or doublet source, for which the disturbing effect decreases with the cube of the distance.

(2) Two-dimensional potential (N. Joukowski) motion with circulation only about a singular point[43] is characterized by a lift proportional to the density, the speed, and the circulation (circuit line-integral of the velocity) itself, provided only that the Kutta-Joukowski condition of finite speed at the trailing edge is fulfilled.

(3) In the two-dimensional discontinuous motion for a flat plate[44] there is a vortex sheet (observed by H. Benard and explained by T. von Kármán for a cylinder)[45] which arises from sheets of velocity discontinuity, although the motion as a whole is irrotational. It is noteworthy that apparently symmetrical causes here do not necessarily produce symmetrical consequences.

For these several types of steady irrotational motion the following questions persist: which solution(s) occurs in nature? which solution is

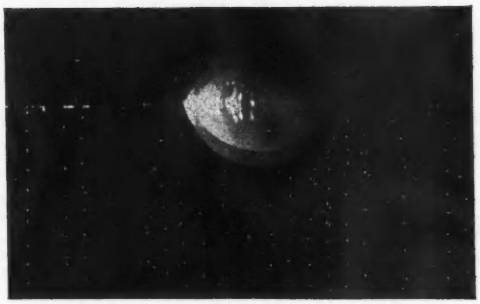


FIG. 7. Bubblegram (second photographs taken after a brief time interval) showing the velocity field about a sphere entering water (G. Birkhoff, Harvard).

[41] L. Prandtl and O. G. Tietjens, *Applied Hydro- and Aeromechanics* (McGraw-Hill Book Company, Inc., New York, 1934).

[42] S. Goldstein, *Modern Developments in Fluid Dynamics* I, II (Oxford University Press, London and New York, 1938).

[43] W. M. Kutta, Illust. Aeron. Mitteilungen 133 (1902). N. E. Joukowski, Z. Flugtechnik Motorluftschiffahrt 1 (1910).

[44] G. Kirchhoff, Crelle's Journal 70, 289 (1869).

[45] H. Bénard, Comptes rend. 147, 970 (1908). T. von Kármán, Gott. Nachr. 509 (1911); 547 (1912). T. von Kármán and H. L. Rubach, Physik. Z. 13, 49 (1912).

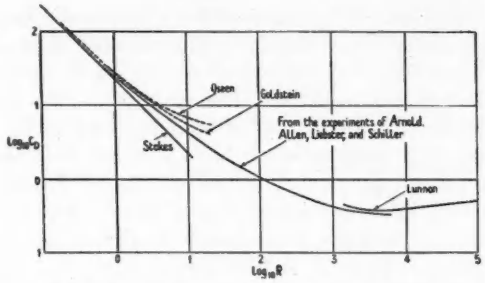


FIG. 8. Goldstein's comparison of theoretical values and experimental data for Stokes' law (see reference 42) for a sphere.

most stable? how does the introduction of viscosity modify the stability?

Potential theory can be well utilized for streamlined bodies also in unsteady motion. As G. Green and G. G. Stokes[46] showed independently for an inviscid incompressible fluid there is virtually an increase in mass owing to the setting of the fluid in motion; for a sphere this so-called "induced mass" is $\frac{1}{2}$ the mass of the displaced fluid. (In general, the virtual mass depends upon the direction of motion, as well as upon the body shape.) Applications of this seemingly abstract concept are rarely given. One example[47] is the motion of a spherical balloon when large differences in fluid density are involved. More striking, however, is the unsteady motion of a weightless sphere in a liquid.[48] The sphere will have an acceleration, relative to an observer, three times that of the liquid itself. Another interesting example occurs in water entry (see Fig. 7), where the variation of cavity drag (dependent upon wake underpressure) may be interpreted as an "impact" induced mass due to the setting of the water in motion.[49]

SPHERE IN A PERFECTLY VISCOUS FLUID

Let us now consider a perfectly viscous fluid for which, by definition, there is a constant coefficient of tangential stress during motion and no slipping along the boundary. The equations of motion were formulated first by Navier.[27] The establishment of the boundary condition, however, was early in doubt with respect to the very question of slipping. Stokes pointed out three possible hypotheses: slipping of fluid along fluid, slipping of molecules, no slipping. He decided in favor of no slipping at all.[27]

The equations of motion for a perfectly viscous fluid were solved by Stokes[50] for the slow steady motion of a rigid sphere, without slipping, in an infinite homogeneous fluid. In the case of such "creeping" motion, the viscous effects are more important than the inertial ones so that "semiquadratic" terms like $(\rho u)\partial u/\partial x$ can be neglected (u, the x velocity component). The resisting force turns out to be proportional to the speed; it can be written

$$F=\frac{24\mu}{\rho U(2a)}(\pi a^2)(\tfrac{1}{2}\rho U^2)=6\pi\mu a U,$$

where a is the radius of the sphere and U the undisturbed stream velocity $(\mu/\rho U 2a = R)$. In this connection one recalls Newton's comment: "That the resistance of bodies is in the ratio of the velocity is more a mathematical hypothesis than a physical one."[51]

Evidently the viscous effects become less significant as one gets farther away from the body so that the inertial terms, in turn, then become relatively more important. For this case C. W. Oseen[52] obtained a better approximate solution

TABLE II. Stokes' law applied to raindrops.

Atmospheric condensations of water	Diameter (cm)	Terminal velocity (cm/sec)	Reynolds number
Fog	0.001	0.30	0.002
Mist	0.01	25	1.67
Drizzle	0.02	75	10
Light rain	0.045	200	60
Moderate rain	0.10	400	267
Heavy rain	0.15	500	500
Excessive rain	0.21	600	840
Cloudburst	0.30	700	1400
(Maximum size)	0.50	800	2667

[46] G. Green, *Math. Papers* 316 (1833) (London, 1871). G. G. Stokes, Camb. Trans. 8 (1843); *Math. and Phys. Papers* (Cambridge, 1880–1905), Vol. 1, p. 17.
[47] G. Birkhoff, *Hydrodynamics* (Princeton University Press, Princeton, 1950).
[48] G. Birkhoff and T. E. Caywood, J. Appl. Phys. 20, 646 (1949).
[49] A. May and J. C. Woodhull, J. Appl. Phys. 21, 1285 (1950).
[50] G. G. Stokes, Trans. Cambridge Phil. Soc. 8, (1843).
[51] I. Newton, *Philosophiae Naturalis Principia Mathematica* (1687), translation by F. Cajori (California, 1946), Book II, Prop. 35, Prob. 7, Cor. V. and Book II, Scholium to Prop. IV.
[52] C. W. Oseen, Arkiv. Math. Astron. Fysik 6, 29 (1910); *Neuere Methoden und Ergebnisse in der Hydrodynamik* (Leipzig, 1927).

for a sphere. Goldstein's[42,53] more complete expression for the drag coefficient is

$$C_D=\frac{24}{R}\left\{1+\frac{3}{16}R-\frac{19}{1280}R^2+\frac{71}{20480}R^3\cdots\right\}.$$

The two theoretical solutions for a sphere differ by 0.05 percent at Reynolds number 0.2, by 1.0 percent at 1.0, and by 10 percent at 5.0; experimentally, the Stokes law can be considered valid for Reynolds numbers less than 0.05, and the Oseen law for Reynolds numbers less than 0.8 (see Fig. 8).

The failure of average texts on theoretical physics to specify generally experimental limitations is probably reflected in statements made in more elementary texts about the applicability of Stokes law. For example, one book states: "In the case of a sphere this resistance is proportional to the radius and to the velocity at moderate speeds. For this reason falling raindrops reach a constant terminal velocity." One immediately wonders what is meant by moderate speeds and what actually is the Reynolds-number range. With respect to the applicability of the Stokes law for the terminal velocity here, Table II, based upon W. J. Humphrey's[54] data, speaks for itself. Humphreys himself warns: "The equations of fall in the case of raindrops of average and large size···are little more than empirical."

One more experimental fact is worth noting here, namely, that for Reynolds numbers greater than 50, small air bubbles in a liquid rise in spherical spirals.[55] For smaller Reynolds numbers D. Riabouchinsky and J. Hadamard[56] showed in 1911 that for creeping motions the Stokes formula has to be modified theoretically for gas bubbles, inasmuch as the boundary condition at the surface of such a sphere specified continuity of velocity across it. Observations,[7] however, indicate that it is the formula for a rigid sphere (with zero fluid velocity on its surface) which actually holds.

[53] S. Goldstein, Proc. Roy. Soc. (London) **A123**, 225 (1929).

[54] W. J. Humphreys, *Physics of the Air* (McGraw-Hill Book Company, Inc., New York, 1940).

[55] H. Nisi and A. W. Porter, Phil. Mag. **46**, 754 (1928); A. W. Porter, *The Method of Dimensions* (Methuen and Company, London, 1933).

[56] D. Riabouchinsky, Bull. Acad. Cracovie 40 (1911); J. Hadamard, Compt. rend. **152**, 1735 (1911).

It was early recognized by I. Newton[51] that for bodies traveling at higher speeds drag is proportional to the square of the speed rather than to the speed itself. For example, in the case of the inelastic impact of fluid particles upon a flat plate transverse to the flow, both the mass impinging per unit time and the change in speed itself are proportional to the speed. Only the front of the plate, of course, is significant in this instance. In 1907, A. G. Eiffel showed that the resistance of two flat plates, together but inclined to the flow, is only 60 percent as much as when they are separated. On the Newtonian picture separation of the plates should have made no difference; accordingly the rear of a plate must have some role to play. In other words, the Newtonian view is adequate for a rarefied medium where the particles are independent, but not for a continuum where the particles are interdependent.

Another general problem of considerable practical importance is the existence of similar flows, which permit the scaling of information obtained experimentally with models. Similar inviscid flows are determined for incompressible fluids by what may be called the Euler number (ratio of the pressure to the so-called dynamic pressure), and for compressible fluids by the Mach number. (The Euler number can be regarded as being inversely proportional to the square of the Mach number for comparative purposes.) An additional parameter, however, is needed for specifying similar viscous flows, namely, the Reynolds number, which measures the inertial influence relative to the viscous effect. The kinematic viscosity is important here. For example, air is effectively 15 times more viscous than water (with respect to kinematic viscosity), although the coefficient of viscosity for air is only 1 percent that for water. Now large kinematic viscosity (low Reynolds number) itself may be said to favor laminar flow inasmuch as it smooths out transversal components of motion, whereas relatively large inertia (high Reynolds number) favors turbulence by continuing transverse components. Thus, an increase in the coefficient of viscosity will increase the speed at which the critical Reynolds number occurs, *ceteris paribus*, but an increase in density will decrease this speed.

Viscous effects, in general, are confined to a small region about the body. In this boundary layer[57] even a small coefficient of viscosity may result in large forces, inasmuch as the latter depend also upon the velocity gradient. If the boundary layer is sucked away with little expenditure of energy, the viscous flow closely approximates the potential flows with and without circulation. "In general, the agreement between theory and experimental results becomes better when viscosity becomes smaller."[41] Much more significant, however, is the indirect effect that even small viscosity may have in selecting stable flow from among the possible types of flow.

Paradoxical descriptions of phenomena have colored the differing outlook of G. Birkhoff in his stimulating book, *Hydrodynamics*.[47] "No treatment of slightly viscous fluids which is based on this theory [classical asymptotic theory of partial differential equations] can come close to describing the real behavior of such fluids. It will never predict a turbulent wake, a vortex sheet, or transition at R_c." A small change in the coefficient of viscosity, for example, from 0.5×10^{-5} to 1×10^{-5}, doubling the Reynolds number, may apparently change the type of flow completely, as from laminar to turbulent flow. He follows Oseen[52] in stressing the fact that the limit of a solution of a partial differential equation for coefficients of the highest order terms approaching zero is not necessarily the same as the solution of the equation with these coefficients already zero. The truth of this statement is evident in that second-order physical effects would otherwise be always negligible.

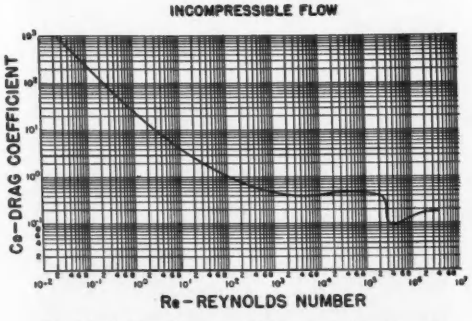


FIG. 9. Sphere drag coefficient as a function of Reynolds number—subsonic speeds.

[57] L. Prandtl, Proc. Math. Congress, Heidelberg (1904).

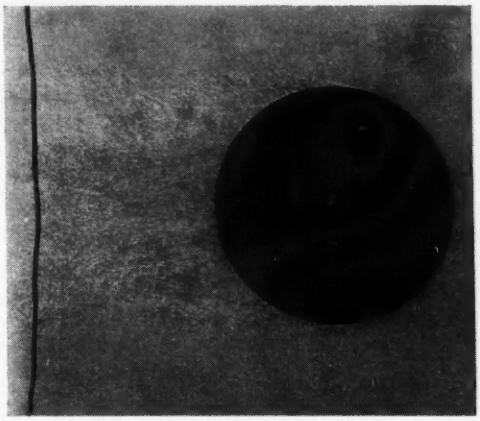


FIG. 10a. Shadowgram of flow about a 37-mm sphere with separation in a laminar boundary layer—Mach number 0.445, Reynolds number 3.68×10^5 (A. C. Charters, BRL free-flight range).

Moreover, no matter how infinitesimal the coefficient of viscosity may become, finite slip flow will certainly not develop in a fluid initially at rest along a boundary. In any event more rigorously thoughtful experimentation is needed before the role of viscosity will be known precisely.

In the light of the above complexity the following naive conclusion from a popular (1934) textbook on theoretical physics exemplifies strikingly how inadequately critical and somewhat meaningless our teaching may be: "The discrepancy [with respect to a circular cylinder] is to be attributed to the fact that even for fluids of low viscosity the friction at the surface is not negligible."

SPHERE IN A REAL FLUID

We shall now consider briefly some particular phenomena of fluids flowing about spheres, first of all, as observed in wind tunnels. G. Eiffel[58] found experimentally in 1911 that the drag coefficient of a sphere was 0.18 as measured in the French tunnel at Auteuil. A year later L. Föppl obtained a value of 0.44 in the German tunnel at Göttingen. It is amusing to find Eiffel explaining away the discrepancy as being due to different critical velocities for the flows, only to have Rayleigh explaining away the latter in view of

[58] G. Eiffel, *Nouvelles Recherches sur la Resistance de l'Air et l'Aviation* (Dounod and Rivat, 1914).

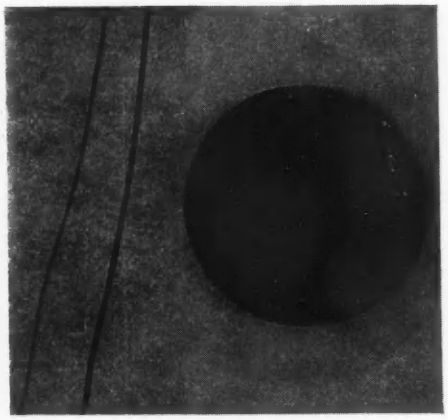


FIG. 10b. Shadowgram of flow about a 37-mm sphere with a separation in a turbulent boundary layer—Mach number 0.519, Reynolds number 4.29×10^5 (A. C. Charters, BRL free flight-range).

the compensating effect of different diameters so that the Reynolds number was unchanged. The real explanation was given by L. Prandtl[59] in 1914 in terms of differences in the turbulence levels of the tunnels. For example, the critical Reynolds number varies from 100,000 in a very turbulent tunnel to 368,000 in a low-turbulence tunnel; in the atmosphere it is about 385,000. The phenomenon associated with these early experiments is sometimes called the Eiffel paradox, namely, that near the critical Reynolds

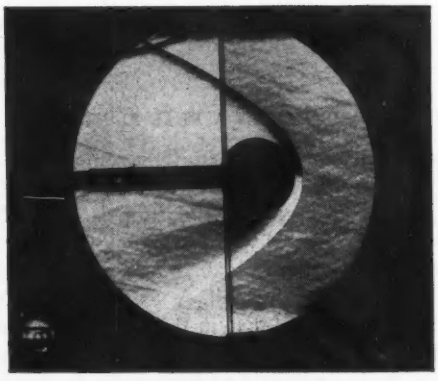


FIG. 11a. Schlieren photograph of flow about a 10-cm sphere—Mach number 3.24, Reynolds number 7×10^5 (R. Lehnert, NOL Aeroballistics Tunnel No. 2).

[59] L. Prandtl, Nachr. Ges. Wiss. Göttingen, Math.-physik Kl. 177 (1914).

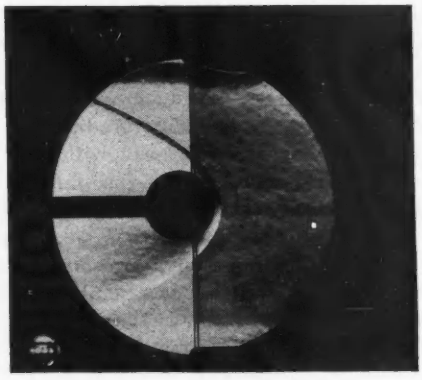


FIG. 11b. Schlieren photograph of flow about a 10-cm sphere—Mach number 4.24, Reynolds number 4×10^5 (R. Lehnert, NOL Aeroballistics Tunnel No. 2).

number the resistance of a sphere actually decreases as its speed increases owing to a large decrease in the drag coefficient (see Fig. 9). The decrease is due to the separation of the fluid from the body farther back in the boundary layer for turbulent than for laminar flow there, with a consequently greater neutralizing (back) pressure at the base [see Figs. 10a and b]. (Turbulence mixing in the stream thickens the boundary layer; although the flow-transition point in the boundary layer may move forward, the outer fluid effectively pulls the boundary fluid farther along the surface to the separation point.)

At supersonic speeds there is a shockwave at the head [see Figs. 11a, b, c], which ac-

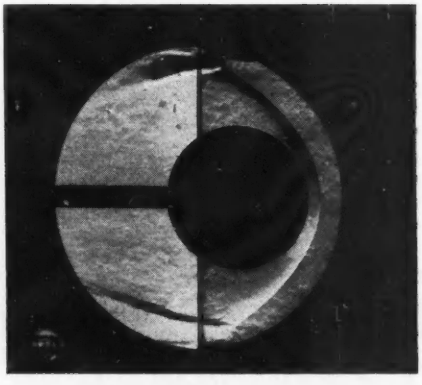


FIG. 11c. Schlieren photograph of flow about a 20-cm sphere—Mach number 5.18, Reynolds number 6×10^5 (R. Lehnert, NOL Aeroballistics Tunnel No. 2).

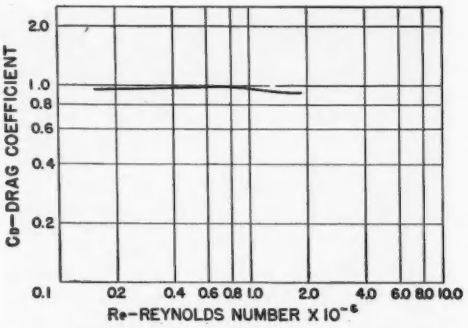


FIG. 12a. Sphere drag coefficient as a function of Reynolds number—supersonic speeds.

FIG. 13. Headwave on rear of projectile due to faster moving muzzle gases (W. Bleakney, Palmer Physical Laboratory).

counts for more than 90 percent of the total drag of a sphere, so that there is little influence resulting from changing Reynolds number [see Figs. 12a and b]. An unusual headwave formed by the faster moving muzzle gas on the back of a projectile is seen in Fig. 13. An interferogram of a sphere in a supersonic jet is shown in Fig. 14. The supersonic flow about a sphere in a low density fluid with slip flow is shown by the nitrogen after-glow technique.

Ballistics ranges[60] afford a complementary means (see Fig. 16) of investigating supersonic phenomena in that the physical conditions of pressure, temperature, and turbulence in free flight are, in general, quite different from those in a wind tunnel. Typical wave phenomena, first photographed with schlieren technique by E. Mach,[61] are evident in the shadowgrams of Figs. 17a, b, c, for different pressures in the Naval Ordnance Laboratory Pressurized Ballistics Range. The wavelets behind the headwave are an indication that supersonic flow exists there. A shock wave occurs also at the separation point, where there is a change of flow direction. At the rear of the body supersonic vortices produce wavelets which coalesce in a shock tailwave, where the vortices merge into large irregular ones. A comparison photograph of a sphere at Mach number 16.7 in monatomic krypton is given in Fig. 18. For diatomic gases a new phenomenon becomes increasingly important as the temperature increases, namely, molecular dissociation. For example, bromine dissociation is found to have occurred already at a Mach number of 13.5 (observed at the Naval Ordnance Laboratory). Such preliminary results in the new

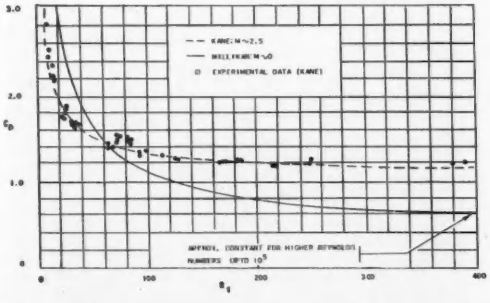


FIG. 12b. Sphere drag coefficient as a function of Reynolds number R_1 computed behind a normal shock wave.

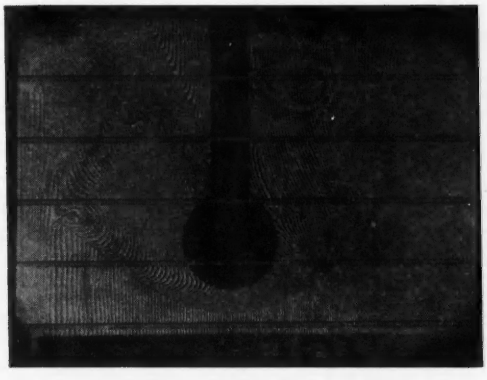


FIG. 14. Interferogram of the flow of an axially symmetric jet about a $\frac{5}{32}$-in. diameter sphere—Mach number 1.7 (R. Ladenburg, Palmer Physical Laboratory).

[60] A. C. Charters and R. N. Thomas, J. Aeronaut. Sci. 12, 468 (1945).
[61] E. Mach, Wien. Ber. 98, 1310 (1889).

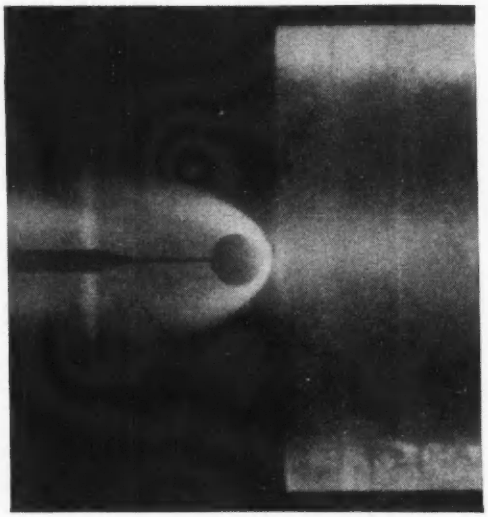


FIG. 15. Nitrogen afterglow photograph of a shock wave for a 0.7 in. sphere in a low density, supersonic flow—Mach number 2.6, Reynolds number 760/inch, $P_{at}=110$ microns (S. A. Schaaf, University of California Low Density Wind Tunnel).

field of aeroballistic research have been described elsewhere by the author in an article entitled, "On Aerophysics Research."[62]

All the above complex phenomena observed in free flight and in wind tunnels give credence to the statements in 1934: "It is a deplorable fact that no theory of drag yet exists which even approximately does justice to the experimental results."[21] Although there have been subsequent improvements, the explanation of the drag of a sphere is still an enigma.

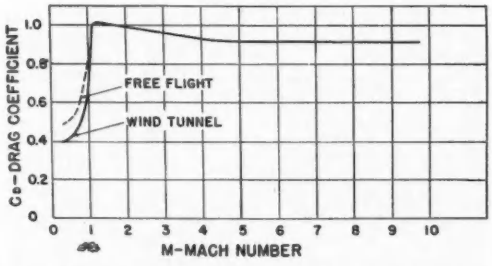


FIG. 16. Comparison of drag-coefficient data from free flight and from wind tunnel for sphere.

[62] R. J. Seeger, Am. J. Phys. 9, 459 (1951).

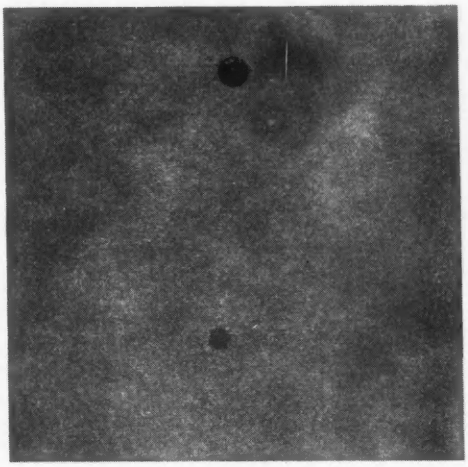


FIG. 17a. Shadowgrams of $\frac{3}{8}$-in. ($\frac{1}{4}$-in.) sphere at Mach number 2.8, Reynolds number 6150 (4200), and pressure 0.0097 atmos (0.0094) (A. R. Witt, NOL Pressurized Ballistics Range No. 3).

FOUNDATIONS OF FLUID DYNAMICS

What is more, many questions persist as to the very foundations of fluid dynamics, for example, with respect to even the equations of motion. In continuum mechanics the conservation equations (five in number) contain time derivatives of two thermodynamic variables (say, pressure and density) and of three kinematic variables (velocity components). In addition, the equations contain components of a stress tensor

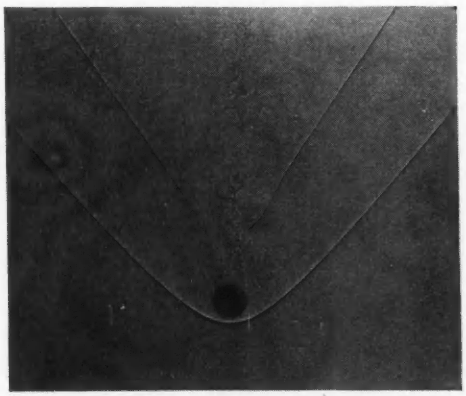


FIG. 17b. Shadowgram of $\frac{1}{2}$-in. sphere at Mach number 2.0, Reynolds number 1×10^6, and pressure 2 atmos (A. R. Witt, NOL Pressurized Ballistics Range No. 3).

FIG. 17c. Shadowgram of ¾-in. sphere at Mach number 1.2, Reynolds number 0.9×10^6, and pressure 4 atmos (A. R. Witt, NOL Pressurized Ballistics Range No. 3).

and of a heat-flow vector. Hence, special assumptions about these quantities have to be added if the equations are to form a determined system. A simple theoretical assumption is the vanishing of both the stress tensor and the heat-flow vector. In this case, the equations of motion become those of Euler for a perfectly inviscid fluid, and the energy equation specifies isentropy along a streamline. A simple phenomenological assumption, on the other hand, is that the stress tensor should be proportional to the velocity gradient, and the heat-flow vector to the temperature gradient. Under the further stipulation of tensor invariance these conditions lead to the Navier-Stokes equations for a perfectly viscous fluid. The viscosity appearing here may be formally classified into the usual shear viscosity and a so-called dilatational viscosity. The second viscosity coefficient is found to be equal to $-(2\mu/3)$ if the average principal stress terms are chosen to agree with the mean static pressure for compressible fluids (this stipulation need not be made). As early as 1851 Stokes recognized the lack of a rigorous justification for this hypothesis.

The only theoretical argument is given by the kinetic theory of a dilute monatomic gas.[63] Yet the derivation of the macroscopic behavior of a gas from its molecular properties is still lacking in rigorous treatment. For example, specifying the initial values of the above five variables of the hydrodynamic equations yields only an approximate solution (not unique) to the fundamental integro-differential equation for the molecular distribution function. Other approximations can generally be obtained by assigning initial values to various moments of the distribution function with respect to the molecular velocity (e.g., in the neighborhood of equilibrium the five-moment approximation of D. Hilbert, and the thirteen-moment extension by H. Grad).[64]

Experimental data are sparse. N. Lieberman's[65] recent measurements of the so-called second coefficient of viscosity apparently explain the acoustic absorption of high frequency waves in liquids. Nevertheless, the fundamental question remains: does the mean pressure depend upon the rate of expansion? Perhaps, the expansion-rate-measuring-apparatus developed for the Naval Ordnance Laboratory by A. M. F. J. Michels will be useful for investigating such phenomena.[66] (It has already been used to show experimentally the constancy of the entropy of a rapidly expanding real gas.)

Turbulence, of course, is still a perennial puzzle. Most success has been achieved to date by regarding it phenomenologically, with assumed ignorance as to the underlying equations of motion. A basic question here is whether or not quasi-steady turbulence is a second solution of the Navier-Stokes equation.[67]

FIG. 18. Shadowgram of ¾-in. sphere shot into krypton at a pressure of 30.5 mm of mercury and at a temperature of 103°K—Mach number 16.7 (C. R. Donaldson, NACA LAL free-flight range).

[63] D. Enskog, Dissertation (Uppsala, 1917). Y. Rocard, *L'hydrodynamique et la théorie cinétique des gaz* (Gauthier-Villars, 1932).

[64] H. Grad, *Lecture Notes on Kinetic Theory and Statistical Mechanics* (New York University, 1950).

[65] L. N. Lieberman, Phys. Rev. 75, 1415 (1949).

[66] A. Siegel, Doctoral dissertation, University of Amsterdam, 1952.

[67] M. Planck, "The Mechanics of Deformable Bodies" (*Introduction to Theoretical Physics* II), (Macmillan Company, New York, 1932).

Even steady-flow phenomena, indeed, are subject to critical analysis. In view of the importance of time variations in determining stability of flow, the following uncomfortable question lurks: does strictly steady flow actually occur in nature?

Finally, the precise role of heat in fluid dynamics opens up a whole new field of questions. In this connection one is reminded of Mach's remarks: "Purely mechanical phenomena do not exist.···Questions concerning the motion of gases cannot be dealt with as questions of pure mechanics, but always involve questions of heat. [Ed. note: Nor can even a thermodynamical treatment always suffice; it is sometimes necessary to go back to the consideration of molecular motions.]"[19] Rapid changes in gases, of course, are associated with temperature changes and expansive forces. Much more startling is P. Duhem's challenging observation: "The supposition of an incompressible viscous fluid is thermodynamically incompatible with the concept of a fluid."[47,68] As flow speeds about Mach number 2.5 are attained, a heat similarity parameter, the Péclet number (measures the relation of convective heat to conductive heat), has to be considered in addition to the Mach number for compressible flow and to the Reynolds number for viscous flow. An unanswered question here is whether these parameters can be regarded as being independent or whether they become involved in a functional relation to describe phenomena at hypersonic (Mach number greater than five) speeds.[62]

TRADITION AND BETRAYAL

In spite of all the above interesting physics, what is the traditional attitude of physicists to fluid dynamics? Perhaps, two statements will suffice. P. Lenard says with respect to the hydrodynamic equations: "They nevertheless contain only old knowledge, nothing fundamentally new is added, or made evident by their treatment."[69] In M. von Laue's recent *History of Physics*,[70] in which the Navier-Stokes equations are not even mentioned, one reads: "Nobody expects these results to go beyond the Newtonian foundations." True—no poem contains new letters, possibly not even unorthodox words—but fresh ideas or possibly even fresh expressions! Pity the poor student who is not brought into contact with the existence of new phenomena in fluid dynamics and the need for new concepts in its physical description. Lest we teachers forget: Tradition does mean basically a handing over; but it also comes from the same etymological source as betrayal. The statement of the mathematician G. Birkhoff must be provocative to us physicists: "It seems surprising that this subject [fluid dynamics], to which so much was contributed by Stokes, Helmholtz, Rayleigh, and other physicists of the nineteenth century, should owe all its more recent progress to mathematicians and engineers. I feel that a renewed interest by physicists in the accurate correlation of theory with experiment, would help greatly in providing it with a more solid scientific basis."[47]

In summary, if the teaching of fluid dynamics in physics involves essentially the understanding of concepts on the part of the student so that he grows intellectually, and if the teaching of fluid dynamics in physics is primarily a matter of selecting the phenomena and their related concepts so that culture develops out of the quality of the overtones, then both in the general physics course of the beginning college student and in the theoretical physics course of the beginning graduate student Shakespeare's words (with respect to the ghost's appearance at the end of Act I) are pertinent:

> Horatio: "O day and night, but this is wondrous strange.
> Hamlet: "And therefore as a stranger give it welcome.
> There are more things in heaven and earth, Horatio,
> Than are dreamt of in your philosophy."

[68] P. Duhem, *Traite d'energetique* 2 (Paris, 1930).
[69] P. Lenard, *Great Men of Science* (Macmillan Company, New York, 1933).
[70] M. von Laue, *History of Physics* (Academic Press, Inc., New York, 1950).

Possible Motions of a Sphere Suspended on a String (The Simple Pendulum)

J. V. HUGHES*
Queen's University, Kingston, Ontario
(Received April 21, 1952)

The possible motions of a sphere suspended on a string are found to consist of combinations of (1) a pendulum type of motion and (2) a rolling motion, in which the ball rolls in angular harmonic motion about an axis very near its center. For (1) it is shown (a) the usual formula for the length of the equivalent simple pendulum should be replaced by $L(1+2r^2/5L^2+4r^3/25L^3+\cdots)$, where r is the radius of the ball and L the distance of point of suspension to center of the sphere, and (b) the best way of starting this motion is to pull the ball aside by a horizontal force through its center. For (2) it is shown that the period of oscillation is very close to $2\pi\sqrt{(2r/5g)}$.

Photographs are given showing experimental verifications of these effects.

I. INTRODUCTION

IT is frequently stated that for a sphere of radius r on the end of a string of length $(L-r)$, swinging as a pendulum, the length of the equivalent simple pendulum is $L(1+\frac{2}{5}r^2/L^2)$. It is not generally realized that this is an approximation. It is accurate for a sphere on the end of a massless rigid rod, where the couple to cause rotation of the sphere is provided by the rigidity of the rod; but for the sphere on a string this couple must be provided by the sphere's weight and the string tension, and consequently the line of the string does not, in general, pass through the center of the sphere.

As the sphere is free to rotate about the point of contact with the string, the system has a degree of freedom more than is assumed for the simple pendulum, and this permits motions other than the simple pendulum motion. The various possible motions are discussed below.

II. BASIC THEORY

The problem of the double pendulum has been treated by Timoshenko,[1] as an example of the Lagrange method using generalized coordinates. In the following the problem is treated by the more direct force methods, as a preliminary to obtaining further results not given explicitly by Timoshenko. The usual assumption is made that all angles are kept sufficiently small for their sines to be replaced by the angles, and their cosines equated to unity.

Consider the system, when swinging, in the position shown in Fig. 1. The coordinates of Q are

$$\left.\begin{aligned} x&=(L-r)\theta+r(\theta+\phi)=L\theta+r\phi,\\ y&=L.\end{aligned}\right\} \tag{1}$$

Resolving vertically and horizontally, and using Eq. (1), one obtains

$$L\frac{d^2\theta}{dt^2}+r\frac{d^2\phi}{dt^2}+g\theta=0. \tag{2}$$

Taking moments about Q and writing mbr^2 for the moment of inertia, where b is a numerical constant ($=\frac{2}{5}$ for a sphere), gives

$$r\frac{d^2\theta}{dt^2}+r\frac{d^2\phi}{dt^2}+\frac{g}{b}\phi=0. \tag{3}$$

Subtracting Eq. (3) from Eq. (2) leaves

$$(L-r)\frac{d^2\theta}{dt^2}+g\theta-\frac{g}{b}\phi=0. \tag{4}$$

Eliminating ϕ between Eqs. (4) and (2) yields a fourth-order differential equation in θ. As we expect harmonic oscillations, we look for solutions of the form $\theta=A\sin(\omega t+\psi)$. Substitution of this in the differential equation gives

$$br(L-r)\omega^4-g(L+br)\omega^2+g^2=0. \tag{5}$$

This is a quadratic in ω^2, with two positive real roots, ω_1 and ω_2. It is convenient to treat the two modes, involving ω_1 and ω_2 respectively, separately. Defining a dimensionless constant α by

* Now at Southwest Research Institute, San Antonio 6, Texas.

[1] S. Timoshenko, *Vibration Problems in Engineering* (Van Nostrand Company, Inc., 1941), p. 203.

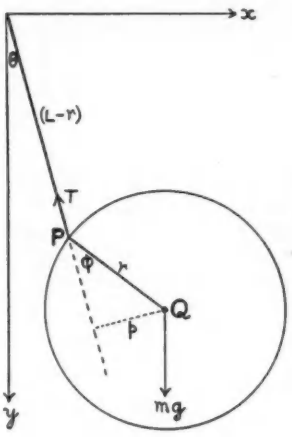


FIG. 1. Coordinate system for the freely swinging pendulum.

$\alpha = r/L$, the solution of Eq. (5) can be written

$$\omega^2 = \frac{g}{2b\alpha L}\frac{1+b\alpha}{1-\alpha}\left\{1 \pm \left(1 - \frac{4b\alpha(1-\alpha)}{(1+b\alpha)^2}\right)^{\frac{1}{2}}\right\}. \quad (6)$$

Since $r < L$, $\alpha < 1$, Eq. (6) can therefore be expanded as a power series in α, giving

$$\text{positive sign: } \omega_1^2 = (g/br)\{1+\alpha+(1+b)\alpha^2 + (1+b+b^2)\alpha^3+\cdots\}; \quad (7)$$

$$\text{negative sign: } \omega_2^2 = (g/L)(1-b\alpha^2 - b^2\alpha^3+\cdots). \quad (8)$$

To obtain the relation between θ and ϕ, Eq. (4) can be rewritten

$$\phi = b\theta + \frac{bL}{g}(1-\alpha)\frac{d^2\theta}{dt^2} = b\theta\left\{1 - \frac{L}{g}(1-\alpha)\omega^2\right\}. \quad (9)$$

III. THE LOW FREQUENCY SOLUTION

Equation (8) gives the period of the pendulum type of motion as $T = 2\pi/\omega_2$. Since the equivalent simple pendulum has a length L_s given by $T = 2\pi\sqrt{(L_s/g)}$, inverting and expanding Eq. (8) gives

$$L_s = L(1+b\alpha^2+b^2\alpha^3+\cdots). \quad (10)$$

This equation shows that the usually quoted formula for the length of the simple pendulum equivalent to a ball on a string is only an approximation, being in fact the first two terms of Eq. (10). The approximation is of course a very good one if $L \gg r$, as is usually the case.

Equation (9), with ω_2 from Eq. (8), yields

$$\phi = b\alpha\theta\{1+b\alpha-b(1-b)\alpha^2+\cdots\}. \quad (11)$$

Thus, if $L \gg r$, $\alpha \ll 1$ and $\phi \ll \theta$, the radius PQ does not move much from the line of the string. This justifies the opening sentence of this section, in which it is stated that this is "pendulum" type of motion.

If the swinging is started at random, it is probable that a component involving ω_1 will be introduced. This is avoided if the angles satisfy Eq. (11) at the moment of release. Suppose the pendulum is drawn aside by a horizontal force, as in Fig. 2.

Resolving vertically and horizontally and taking moments round P, one obtains

$$f = r\phi/\theta. \quad (12)$$

To satisfy Eq. (11), retaining only the first power of α, therefore requires $f = rb\alpha$. Thus, the horizontal force should be applied below the center of the ball by an amount $2r^2/5L$. For $L \gg r$, a very good approximation to this will be produced by pulling the ball aside by a horizontal force through its center. If the force were applied at P, or at the bottom of the ball, then $f = \pm r$, and from Eq. (12) this would give $\phi = \pm\theta$. Consequently both these methods would introduce a large amplitude of the unwanted component.

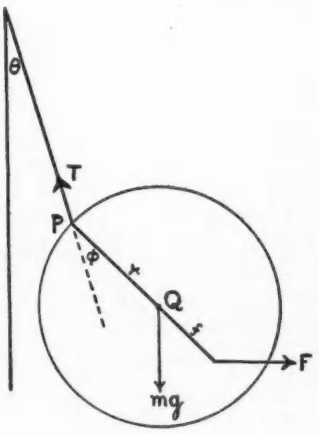


FIG. 2. System of forces for pendulum pulled aside, before releasing.

IV. THE HIGH FREQUENCY SOLUTION

Equation (7) gives the period of oscillation as

$$T=2\pi\left\{\frac{br}{g}(1-\alpha+\cdots)\right\}^{\frac{1}{2}}\simeq 2\pi\left(\frac{2}{5}\frac{r}{g}\right)^{\frac{1}{2}}.$$

Equation (9) gives

$$\phi=b\theta(1-L/br)=-\theta(1-b\alpha)/\alpha. \qquad (13)$$

It is seen that ϕ is opposite in sign to θ, and larger. The ball "rolls," with relatively little angular motion of the string. To determine the center of rotation, consider a point whose rest coordinates on Fig. 1 are $(0, L+Y)$, so that the point is Y below the center of the sphere. When swinging, its x coordinate will be

$$x=(L-r)\theta+(r+Y)(\theta+\phi)=L\theta+\alpha L\phi+Y(\theta+\phi).$$

Thus, $x=0$, if

$$Y=-L(1+\alpha\phi/\theta)/(1+\phi/\theta).$$

Substitution from Eq. (13), retaining only the lowest power of α, yields

$$Y=\alpha^2 Lb=br(r/L).$$

Thus, the ball will roll about a point only slightly below the center of the sphere.

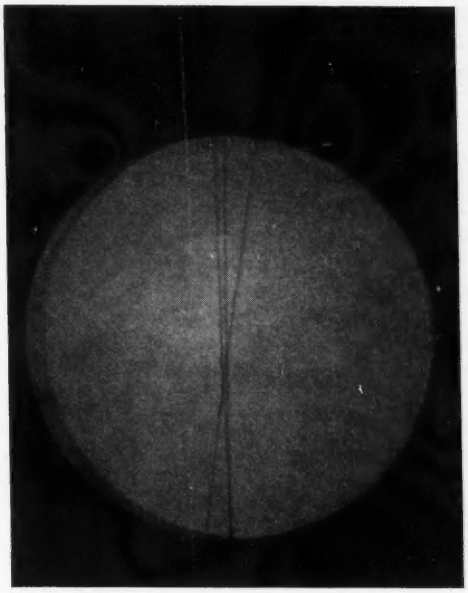


FIG. 4. Triple exposure photograph of bob undergoing rotational oscillations.

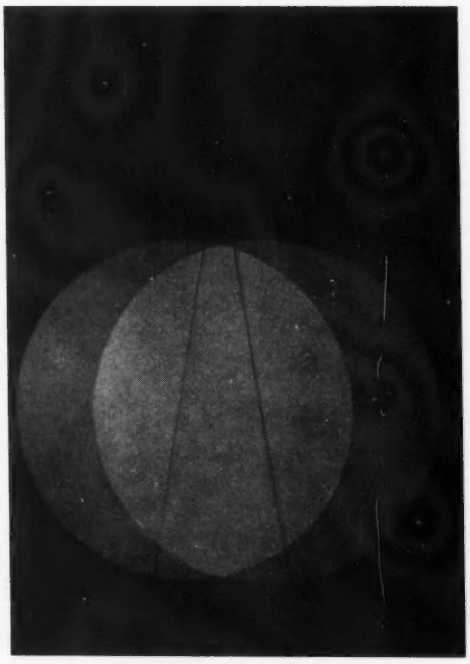


FIG. 3. Double exposure photograph of simple pendulum motion.

V. EXPERIMENTAL VERIFICATION

For experimental work, a flat disk ($b=0.5$) was used, since the effects should be more pronounced with the larger b. The disk was painted white, and the diameter PQ was marked in black. The disk was 9 inches in diameter.

Results of Sec. III (pendulum motion). For this system a string 3 inches long was used. Thus, $r=4\frac{1}{2}$ inches; $L=7\frac{1}{2}$ inches; $\alpha=0.6$; $b=0.5$.

The system was set swinging and photographed, two exposures of 1/100 sec being made on the same film. The photograph is reproduced in Fig. 3. Application of a straight edge will show that PQ is not in line with the string and that the angle ϕ has the same sign as θ. The angle between the two string positions was measured and called θ'; and the angle between the two diameters was called $(\theta'+\phi')$. The measurements gave $\theta'=13.7°$ and $\theta'+\phi'=18.1°$; thus,

experimentally, $\phi/\theta=0.32$. From Eq. (11), theoretically $\phi/\theta=0.36$. The agreement between experiment and theory is therefore reasonably good.

An attempt was made to check the period. The usual formula for the equivalent simple pendulum gives $T=0.948$ sec, while Eq. (10) gives 0.975 sec. Owing to difficulty in keeping the pendulum swinging in one plane, the experimentally determined periods varied from 0.960 to 0.970 sec. The result is therefore not conclusive, but tends to favor the theory given here.

Results of Sec. IV (rolling motion). For this a string 9 inches long was used. Thus, $r=4\frac{1}{2}$ inches; $L=13\frac{1}{2}$ inches; $\alpha=\frac{1}{3}$; $b=0.5$.

Three exposures, at different instants of the motion, were taken on the same film, and the photograph is reproduced in Fig. 4. It was difficult to start the rolling motion without some component of the pendulum motion also occurring, which accounts for the fact that the three diameters do not intersect at a point. Nevertheless, it can be seen that the center of rotation is on the disk, slightly below the center of the disk.

The period was measured as 0.36 to 0.38 sec (counting such fast oscillations was difficult). Equation (7) gives a period of 0.384 sec. The agreement is as good as can be expected.

VI. ACKNOWLEDGMENTS

It is a pleasure to thank Dr. H. W. Harkness for checking the theory, and Dr. E. E. Watson for assistance with the photography.

The Tape Recordings of Important Speeches

BY

E. U. CONDON, K. K. DARROW, E. FERMI, J. C. SLATER

Free tape recordings are available on loan to members of the Association who wish to use them for presentation to classes, seminars, or science clubs. The subjects are four of the six lectures presented as a Symposium on Physics Today at the Twentieth Anniversary meeting of the American Institute of Physics, Chicago, Illinois, on October 25, 1951, as follows:

The atom. E. U. CONDON.
Physics as science and art. K. K. DARROW.
The nucleus. E. FERMI.
The solid state. J. C. SLATER.

Professor Slater's speech has been added to the list since our last announcement.

Recordings are now available in two forms:

1. *Double track.* Each speech is complete on one reel, which must be played at $3\frac{3}{4}$ inches per sec. No rewinding is necessary. Track No. 1 is played to the end, then the reels are reversed (and turned over) to play track No. 2. This procedure rewinds the tape into its original condition.
2. *Single track.* Two 7-inch reels are required for each address. They must be played at $7\frac{1}{2}$ inches per sec, and, of course, rewound to restore them to their original condition.

Requests for recordings should be made to THOMAS H. OSGOOD, Editor, *American Journal of Physics*, Michigan State College, East Lansing, Michigan. To allow some flexibility in scheduling, a request should specify a preferred date and two alternative acceptable dates, as well as the type of recording (single or double track, or either).

No charge is made for the loan of recordings. The borrower is expected, however, to pay the return postage to the Editor's office or to any other address the Editor specifies. Transportation is to be by first class mail, special delivery, which will cost less than one dollar per reel at present rates. Reels will be mailed in convenient reusable boxes.

Prospective borrowers may consult the published papers in *Physics Today*, November, 1951, January, 1952, and March, 1952.

Reproductions of Prints, Drawings, and Paintings of Interest in the History of Physics

50. Fluents and Fluxions

E. C. Watson
California Institute of Technology, Pasadena 4, California

(Received April 21, 1952)

The frontispiece of John Colson's *The Method of Fluxions and Infinite Series* together with explanatory discussion is reproduced. It is an interesting attempt to explain the method of the calculus pictorially.

THE method of fluxions (which we now call the infinitesimal calculus) apparently occurred to Newton while he was still an undergraduate at Cambridge (he took his B.A. degree in January, 1665, shortly after his twenty-second birthday). The *Methodus Fluxionum* was begun in 1664 and finished in 1671, but remained unpublished until 9 years after his death, when in 1736 it appeared in an English translation by John Colson as *The Method of Fluxions and Infinite Series*. The Latin original did not appear until 1779, when it was published in the first volume of the *Opera Omnia*. In the meantime, however, the fundamental principles of the method had been published in Lemma II in the second book of the *Principia* (1687), in the *Opticks* of 1704, and elsewhere.

Colson's translation of 1736 is of especial interest from the point of view of this series of reproductions because of the frontispiece. This engraving is here reproduced and shows, as an illustration of the method of fluxions, a hunter attempting to shoot down two birds at once.

The full title of this important work reads as follows: *The Method of Fluxions and Infinite Series; with its Application to the Geometry of Curve-Lines. By the Inventor, Sir Isaac Newton, Kt, Late President of the Royal Society. Translated from the Author's Latin Original not yet made publick. To which is subjoin'd, A Perpetual Comment upon the whole Work, Consisting of Annotations, Illustrations, and Supplements, In order to make this Treatise A compleat Institution for the use of Learners.* By John Colson, M.A. and F.R.S., Master of Sir Joseph Williamson's free Mathematical-School at Rochester. London: Printed by Henry Woodfall; and Sold by John Nourse, at the Lamb without Temple-Bar. MDCCXXXVI.

Colson's translation of Newton's definitions of the terms "Fluents" and "Fluxions" reads as follows:

"Now those Quantities which I consider as gradually and indefinitely increasing, I shall hereafter call *Fluents*, or *Flowing Quantities*, and shall represent them by the final Letters of the Alphabet v, x, y, and z; that I may distinguish them from other Quantities, which in Equations are to be consider'd as known and determinate, and which therefore are represented by the initial Letters a, b, c, etc. And the Velocities by which every Fluent is increased by its generating Motion, (which I may call *Fluxions*, or simply Velocities or Celerities,) I shall represent by the same Letters pointed thus $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$. That is, for the Celerity of the Quantity v I shall put $\dot{v}$, and so for the Celerities of the other Quantities x, y, and z, I shall put $\dot{x}$, $\dot{y}$, and $\dot{z}$ respectively."

The frontispiece is explained as follows:

"But lastly, I believe it may not be difficult to give a pretty good notion of Fluents and Fluxions, even to such Persons as are not much versed in Mathematical Speculations, if they are willing to be inform'd, and have but a tolerable readiness of apprehension. This I shall here attempt to perform, in a familiar way, by the instance of a Fowler, who is aiming to shoot two Birds at once, as is represented in the Frontispiece. Let us suppose the right Line AB to be parallel to the Horizon, or level with the Ground, in which a Bird is now flying at G, which was lately at F, and a little before at E. And let this Bird be conceived to fly, not with

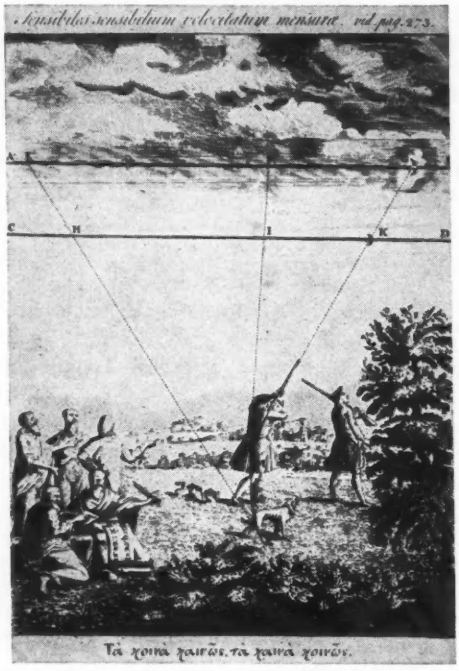


FIG. 1. A hunter attempting to shoot down two birds at once, as an illustration of the method of fluxions. (From John Colson's English translation of Newton's *Methodus Fluxionum*, 1736.)

an equable or uniform swiftness, but with a swiftness that always increases, (or with a Velocity that is continually accelerated,) according to some known rate. Let there also be another right Line *CD*, parallel to the former, at the same or any other convenient distance from the Ground, in which another Bird is now flying at *K*, which was lately at *I*, and a little before at *H*; just at the same points of time as the first Bird was at *G*, *F*, *E*, respectively. But to fix our Ideas, and to make our Conceptions the more simple and easy, let us imagine this second Bird to fly equably, or always to describe equal parts of the Line *CD* in equal times. Then may the equable Velocity of this Bird be used as a known measure, or standard, to which we may always compare the inequable Velocity of the first Bird. Let us now suppose the right Line *EH* to be drawn, and continued to the point *L*, so that the proportion (or ratio) of the two Lines *EL* and *HL* may be the same as that of the Velocities of the two Birds, when they were at *E* and *H* respectively. And let us farther suppose, that the Eye of a Fowler was at the same time at the point *L*, and that he directed his Gun, or Fowling-piece, according to the right Line *LHE*, in hopes to shoot both the Birds at once. But not thinking himself then to be sufficiently near, he forbears to discharge his Piece, but still pointing it at the two Birds, he continually advances towards them according to the direction of his Piece, till his Eye is presently at *M*, and the Birds at the same time in *F* and *I*, in the same right Line *FIM*. And not being yet near enough, we may suppose him to advance farther in the same manner, his Piece being always directed or level'd at the two Birds, while he himself walks forward according to the direction of his Piece, till his Eye is now at *N*, and the Birds in the same right Line with his Eye, at *K* and *G*. The Path of his Eye, described by this double motion, (or compounded of a progressive and angular motion,) will be some Curve-line *LMN*, in the same Plain as the rest of the figure, which will have this property, that the proportion of the distances of his Eye from each Bird, will be the same every where as that of their respective Velocities. That is, when his Eye was at *L*, and the Birds at *E* and *H*, their Velocities were then as *EL* and *HL*, by the Construction. And when his Eye was at *M*, and the Birds at *F* and *I*, their Velocities were in the same proportion as the Lines *FM* and *IM*, by the nature of the Curve *LMN*. And when his Eye is at *N*, and the Birds at *G* and *K*, their Velocities are in the proportion of *GN* to *KN*, by the nature of the same Curve. And so universally, of all other situations. So that the Ratio of those two Lines will always be the sensible measure of the ratio of those two sensible Velocities. Now if these Velocities, or the swiftnesses of the flight of the two Birds in this instance, are call'd *Fluxions;* then the Lines described by the Birds in the same time, may be call'd their *contemporaneous Fluents;* and all instances whatever of Fluents and Fluxions, may be reduced to this Example, and may be illustrated by it.

"And thus I would endeavour to give some notion of Fluents and Fluxions, to Persons not much conversant in the Mathematicks; but such

as had acquired some skill in these Sciences, I would thus proceed farther to instruct, and to apply what has been now deliver'd. The contemporaneous Fluents being $EF=y$, and $HI=x$, and their rate of flowing or increasing being suppos'd to be given or known; their relation may always be express'd by an Equation, which will be compos'd of the variable quantities x and y, together with any known quantities. And that Equation will have this property, because of those variable quantities, that as FG and IK, EG and HK, and infinite others, are also contemporaneous Fluents; it will indifferently exhibit the relation of those Lines also, as well as of EF and HI; or they may be substituted in the Equation, instead of x and y. And hence we may derive a Method for determining the Velocities themselves, or for finding Lines proportional to them. For making $FG=Y$, and $IK=X$; in the given Equation I may substitute $y+Y$ instead of y, and $x+X$ instead of x, by which I shall obtain an Equation, which in all circumstances will exhibit the relation of those Quantities or Increments. Now it may be plainly perceived, that if the Line MIF is conceived continually to approach nearer and nearer to the Line NKG, (as just now, in the instance of the Fowler,) till it finally coincides with it; the Lines $FG=Y$, and $IK=X$, will continually decrease, and by decreasing will approach nearer and nearer to the Ratio of the Velocities at G and K, and will finally vanish at the same time, and in the proportion of those Velocities, that is, in the Ratio of GN to KN. Consequently in the Equation now form'd, if we suppose Y and X to decrease continually, and at last to vanish, that we may obtain their ultimate Ratio; we shall thereby obtain the Ratio of GN to KN. But when Y and X vanish, or when the point F coincides with G, and I with H, then it will be $EG=y$, and $HK=x$; so that we shall have $\dot{y}:\dot{x}::GM:KN$. And hence we shall obtain a Fluxional Equation, which will always exhibit the relation of the Fluxions or Velocities, belonging to the given Algebraical or Fluential Equation.

"Thus, for Example, if $EF=y$, and $HI=x$, and the indefinite Lines y and x are supposed to increase at such a rate, as that their relation may always be express'd by this Equation $x^3-ax^2+axy-y^3=0$; then making $FG=Y$, and $IK=X$, by substituting $y+Y$ for y, and $x+X$ for x, and reducing the Equation that will arise, (see before, pag. 255).) we shall have $3x^2X+3xX^2+X^3-2axX-aX^2+axY+aXy+aXY-3y^2Y-3yY^2-Y^3=0$, which may be thus express'd in an Analogy, $Y:X::3x^2-2ax+ay+3xX+X^2-aX:3y^2-ax-aX+3yY+Y^2$. This Analogy, when Y and X are vanishing quantities, or their ultimate Ratio, will become $Y:X::3x^2-2ax+ay:3y^2-ax$. And because it is then $Y:X::GN:KN::\dot{y}:\dot{x}$, it will be $\dot{y}:\dot{x}::3x^2-2ax+ay:3y^2-ax$. Which gives the proportion of the Fluxions. And the like in all other cases. Q. E. I."

An Electronic Differential Analyzer

KEITH R. SYMON AND ROBERT P. POPLAWSKY*
Wayne University, Detroit, Michigan
(Received April 10, 1952)

An electronic differential analyzer designed primarily for classroom teaching is described. The value of building such a device as a project for advanced students is emphasized. The method of setting up differential equations on the differential analyzer is explained, and a few sample solutions are illustrated.

THE present article describes briefly a small electronic differential analyzer built at Wayne University. The project had two principal objectives: first, to provide an instructive experience in the design and construction of electronic apparatus; and second, to produce at a minimum cost in time and money a machine which would be a useful teaching aid in courses in intermediate and advanced physics, and which might also be useful in solving certain kinds of research problems.

* Now with University of Michigan Research Institute.

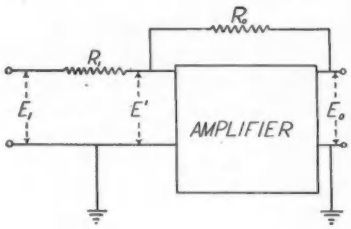


FIG. 1. Computing amplifier connected to multiply by a constant.

As to the first objective, the project has been eminently successful. There is no substitute for first-hand experience in working out a design, building a machine, and then spending the weeks or months required to get the flaws out of it until it runs as planned. As to the second objective, we can claim moderate success. The machine was designed, built, and put in working order by one of us (R.P.P.) as a master's thesis project, representing half-time work for one semester, under the guidance and supervision of one or two staff members. At present, it is capable of solving linear differential equations with constant coefficients or sets of such equations up to about the seventh order, with a design accuracy of about 1 percent for each mathematical operation performed. Its performance is continuously being improved, and is already entirely satisfactory for classroom demonstration. Additional components which will permit the solution of nonlinear differential equations and equations with variable coefficients are under development as a second master's thesis project. The total cost of materials, including parts for additional components to be built, has been $244.00.

It is the purpose of this article to call to the attention of physics teachers the desirability of initiating projects of this type. We hold no brief for the superiority of the particular design at which we have arrived. On all questions of design, we made those decisions which seemed to us to offer the greatest likelihood of leading to something workable in the shortest time with the least expense, rather than those decisions which might have lead ultimately to the highest degree of perfection.

PRINCIPLE OF THE COMPUTING AMPLIFIER

The principle of operation of the computing amplifier has been explained by many authors.[1]

[1] E.g., Ragazzini, Randall, and Russell, Proc. Inst. Radio Engrs. **35**, 444 (1947).

In Fig. 1, the computing amplifier has a large negative gain $-A$, so that

$$E_0 = -AE', \tag{1}$$

where E_0 and E' are the voltages across the indicated points in the circuit. Assuming that the amplifier has an infinite input impedance, the currents through resistors R_1 and R_0 are equal:

$$(E_1 - E')/R_1 = (E' - E_0)/R_0. \tag{2}$$

Substituting E' from Eq. (1) in Eq. (2), and solving for E_0, we obtain

$$E_0 = -(R_0/R_1)E_1[1+(1/A)(1+R_0/R_1)]^{-1}. \tag{3}$$

If A is greater than 1000, and if R_0/R_1 is less than 10, then within an accuracy of about 1 percent,

$$E_0 = -(R_0/R_1)E_1. \tag{4}$$

Thus the computing amplifier may be used as in Fig. 1 to multiply an input voltage E_1 by a constant negative factor.

Figure 2 shows a computing amplifier used as an adder. Applying a similar analysis to that given in the preceding paragraph, we obtain

$$E_0 = -\{(R_0/R_1)E_1+(R_0/R_2)E_2+(R_0/R_3)E_3\} \times\{1+(1/A)[1+(R_0/R_1) +(R_0/R_2)+(R_0/R_3)]\}^{-1}, \tag{5}$$

or, to a close approximation,

$$E_0 = -\frac{R_0}{R_1}E_1-\frac{R_0}{R_2}E_2-\frac{R_0}{R_3}E_3. \tag{6}$$

Similarly, for Fig. 3, to a close approximation if A is large, we have

$$E_0 = -\frac{1}{R_1C_0}\int_0^t E_1 dt, \tag{7}$$

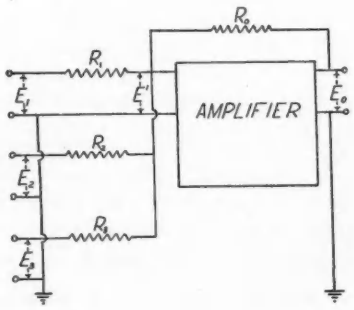


FIG. 2. Computing amplifier connected as adder.

where it is assumed that at $t=0$ the condenser C_0 is uncharged. It is clear that one can combine the operations of addition and integration in one computing amplifier circuit. In principle, a computing amplifier can be made to differentiate by using a feed-in condenser and feedback resistor, but differentiators are to be avoided, as they tend to amplify any hum or stray noise present.

With computing amplifiers performing the basic operations of addition, multiplication by a constant, sign reversal, and integration, any homogeneous linear ordinary differential equation with constant coefficients can be solved. Examples will be given in a subsequent section.

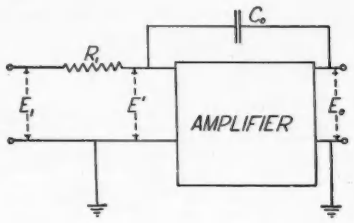


FIG. 3. Computing amplifier connected as integrator.

DESIGN OF THE COMPUTING AMPLIFIER

Several important and often conflicting considerations enter into the design of the computing amplifier. Perhaps the first decision to be made is the time required to complete a solution. A distinction is usually made between high speed computers which complete a solution in a few milliseconds, and low speed computers which complete a solution in a few seconds to several minutes.[2] With the former, the solution is repeated many times per second and presented on the face of a cathode-ray tube, while with the latter, the solution is usually drawn by a recording voltmeter. Partly from perversity, and because we could see a number of possible advantages, we chose a solution time of one-tenth second, midway between the high and low computing speeds ordinarily used. Since our computer is to be used for classroom demonstration, it seemed desirable to use a cathode-ray tube presentation. A solution requiring one-tenth second repeated five times per second on a cathode-ray tube with a long persistence screen gives a satisfactory presentation. By using a relatively long solution time, we hoped to avoid some of the high frequency response difficulties which arise with high speed computers. Also, at this speed, it is possible to use relays to control the computing cycle.

The greater the gain of the computing amplifier, the smaller will be the error in each computing operation, as shown in the preceding section. However, increasing the gain makes other design problems more severe. In our design, we decided on a gain of 1000, corresponding to an error of about 1 percent in each computing operation with allowable multiplication factors no greater than 10 in any one operation. The actual amplifier gain as designed turned out to be about 1200.

The frequency response of a computing amplifier must be very carefully controlled. At low frequencies, the gain must not fall appreciably at least down to the solution repetition frequency, or $1/T$, where T is the solution time. A dc amplifier would meet this requirement, but dc amplifiers are subject to zero drift difficulties. We decided to use resistance-coupled amplifiers with an RC time constant of 2.5 seconds. The high frequency response is determined by the maximum time rate of change of the functions to be handled by the computer. We selected 1 percent of the solution time, or 1 millisecond, as the minimum time within which a significant change in any function would be allowed. A frequency response up to 1000 cycles per second would seem to allow the computing amplifiers to handle functions varying at the maximum allowed rate. However, detailed analysis shows that when the computer is generating oscillating solutions of a differential equation, any falling off in high frequency response of integrators, adders, and constant multipliers results in a progressive increase in amplitude of the oscillating voltage.[3] For example, in order to generate a sine wave of 1000 cycles per second with the circuit of Fig. 9, with an increase in amplitude of no more than 1 percent in 0.1 second, the upper frequency limit of the amplifiers at the half-power point must be at least 150 kilocycles per second. We based our amplifier design on a half-power point of about 20 kilocycles per second, which should allow generation of a sine wave at

[2] J. L. Ryerson, Am. J. Phys. **19**, 90 (1951).

[3] A. B. MacNee, Proc. Inst. Radio Engrs. **37**, 1315 (1949).

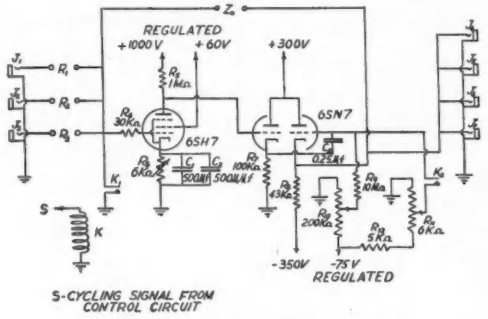


FIG. 4. Computing amplifier circuit.

300 cycles per second with a 1 percent increase in amplitude in 0.1 second.

Since a computing amplifier is used with a feedback resistor or condenser connected directly from output to input, its phase shift characteristics are very important. Within the band of frequencies required for computation the phase shift is nearly 180°, or an odd multiple thereof, and the feedback is negative. The precise conditions for stability of a feedback amplifier are rather involved, but roughly we can say that if the phase shift through the amplifier and feedback path becomes 0° or 360° at any frequency at which the amplifier gain is greater than or equal to one, the amplifier will oscillate. This implies that the phase shift must be controlled not only within the band of frequencies required for computation, but throughout the entire band where the gain is more than one. It can be shown[4] that if the phase shift is not to differ by more than 180° from the phase shift at mid-frequencies, the gain of the amplifier must not fall off at high frequencies more rapidly than 12 db per octave, (that is, when the frequency is doubled, the voltage gain must not decrease by more than a factor of four). This means that if the gain is 1000 at 20 kilocycles, the gain can not drop below one up to at least 600 kilocycles. Since a resistance-coupled amplifier with one *RC* network per stage will have an additional phase shift at high frequencies of 90° per stage, a feedback amplifier of more than two stages is inherently unstable unless special steps are taken to keep the phase shift small. After a few experiments with three-stage amplifiers, we decided to try to eliminate the phase shift problem by using only one stage of amplification.

Our present amplifier circuit is shown in Fig. 4. In it J_1, J_2, and J_3 are input phone jacks leading into the amplifier through double banana jacks into which can be plugged the computing resistors R_1, R_2, and R_3, (Fig. 2). Also, J_4, J_5, J_6, and J_7 are output phone jacks. A double banana jack is connected between input and output for insertion of the feedback resistor or condenser Z_0. The amplifying stage is a single 6SH7 tube which gives a voltage gain of about 1400 in the circuit shown. The two cathode follower stages provide a high impedance load for the 6SH7, a low impedance output, and a means for automatically zeroing the amplifier before each computing cycle. The over-all gain of the amplifier is about 1200, and the amplifier will deliver over 100 volts positive or negative output to the load of 25,000 ohms. Resistance values of 100,000 ohms to 10 megohms are used in computing. In operation, the relay contacts K_1 and K_2 are closed for 0.1 second before each computation begins. Contact K_1 grounds the amplifier input, and contact K_2 connects the grid of the last cathode follower to a tap on R_{11} which is adjusted so that the amplifier output voltage is then exactly zero. Condenser C_3 is charged up during this 0.1 second to the proper voltage so that when K_1 and K_2 are opened, the amplifier output will be zero whenever the input is zero, during the 0.1 second required to complete a solution. Resistor R_6 is adjusted so that the voltage at the cathode of the first cathode follower is 150 volts. Resistor R_{10} is adjusted to compensate for the leakage through C_3 so that with the amplifier input grounded and with K_2 open, the output remains at zero without drifting for a period of several seconds. Unfortunately, the amplifiers showed a slight tendency to oscillate when feedback was applied through Z_0, and this has been corrected temporarily by adding a 500 $\mu\mu$f condenser between the plate of the 6SH7 tube and ground. This has no apparent effect for most applications we have made of the computer, but it will of course reduce its accuracy in handling rapidly oscillating functions. It is hoped, when time permits, that a less drastic expedient can be worked out to eliminate the oscillations, which,

[4] F. E. Terman, *Radio Engineering* (McGraw-Hill Book Company, Inc., New York, 1947), third edition, p. 316.

incidentally, were not present in the original breadboard model—a kind of experience shared by everyone who has tried to design electronic circuits.

Our differential analyzer makes use of ten computing amplifiers like that shown in Fig. 4, built on a 13 in.×17 in. chassis behind a standard $8\frac{3}{4}$ in. high relay rack panel. The panel arrangement is shown in Fig. 5.

CONTROL CIRCUIT

Figure 6 is a diagram of the control circuit which regulates the computing cycle. In automatic operation, a multivibrator alternately energizes and relaxes relay M. When relay M is energized, a 45-volt dc cycling signal is sent through contact M_1 to each amplifier, energizing each relay K (Fig. 4). Relay K returns the amplifier input and output to zero. When relay M relaxes, the cycling signal is removed, relay K in each amplifier opens, and the amplifiers are ready to compute. The solution time can be adjusted from 0.1 second to about 1 second by adjusting R_3. The off time during which the amplifiers are returned to zero is about 0.1 second.

The cycling signal also controls relay N, which supplies a regulated voltage of 100 volts to phone jacks J_1–J_{10} during the computing part of the cycle. This 100 volts is the standard dc reference voltage used as a basis for all computations. All other dc voltages required are generated by computing amplifiers (Fig. 1) as constant submultiples of this reference voltage. Relay N operates in such a way that jacks J_1–J_{10} are connected to +100 volts about 0.5 millisecond after contacts K_1 and K_2 are opened on all

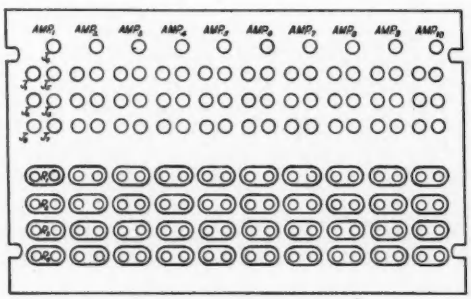


FIG. 5. Computing amplifier panel.

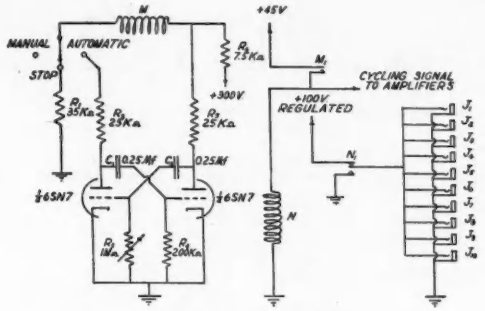


FIG. 6. Control circuit.

amplifiers. Thus, for about 0.5 millisecond all amplifiers remain at zero output after K_1 and K_2 are open, and then the 100-volt reference voltage is applied simultaneously at appropriate points in the computing circuit and computation begins. This arrangement insures that all computations shall begin simultaneously, independently of slight differences in time of operation of relays K in the various amplifiers.

Switch S_1 permits either manual or automatic control of the computing cycle. For manual control, S_1 is placed in the "Stop" position; this closes relay M and allows the amplifiers to come to zero. Switch S_1 is then placed in the "Manual" position; this opens relay M and allows computation to proceed.

Any desired variable in a problem can be plotted horizontally or vertically on the face of a cathode-ray tube by connecting the deflection amplifier to the computing amplifier output at which that variable is to be found. We use a 5CP7 tube with a long persistence screen. The deflection amplifiers are dc amplifiers with a variable gain up to a maximum of about two. In most cases a variable is to be plotted against the time t which can be obtained most conveniently from an integrating amplifier (Fig. 3) as

$$t=0.01\int_0^t 100dt. \tag{8}$$

By plotting t from Eq. (8) horizontally, we achieve exact synchronization of the sweep with the computing cycle. By using a relay, operated by a trigger circuit, triggered by the cycling signal, it is possible to plot two variables during alternate cycles so that both are visible together on the screen.

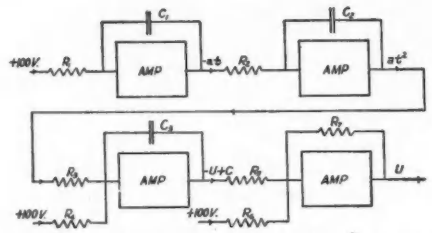


FIG. 7. Circuit for computing $U=-C+\int_0^t(at^2+b)dt$. ($R_1C_1=100/a$, $R_2C_2=\frac{1}{2}$, $R_3C_3=1$, $R_4C_4=1/b$, $R_7/R_5=1$, and $R_7/R_6=C$).

The precision of the computer during automatic cycling is quite adequate. For all ordinary problems, the solution repeats itself from one cycle to the next as nearly as can be seen on the (long persistence) oscilloscope screen.

SETTING UP THE COMPUTER TO SOLVE PROBLEMS

As a first example, let us take the differential equation

$$dU/dt=at^2+b, \qquad U(t=0)=-C. \tag{9}$$

In order to write Eq. (9) in a form suitable for the differential analyzer, we integrate to obtain

$$U=-C+\int_0^t(at^2+b)dt. \tag{10}$$

To generate the function at^2, we write

$$at=0.01a\int_0^t 100dt, \tag{11}$$

$$at^2=2\int_0^t atdt. \tag{12}$$

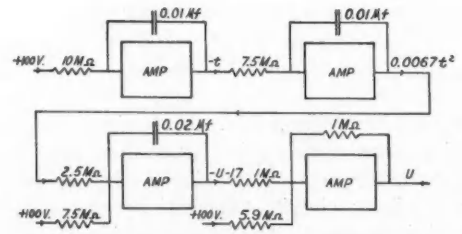


FIG. 8. Circuit for computing $U=-17+0.02\times\int_0^t(0.0067t^2+33)dt$.

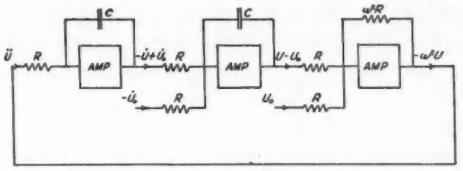


FIG. 9. Circuit for solving $\ddot{U}+\omega^2U=0$.

A circuit for solving Eq. (9) can then be arranged as in Fig. 7.

We have so far omitted all reference to units and scale factors. Before Eq. (9) can be set up on the differential analyzer, all terms must be expressed in volts, and t must be expressed in some convenient unit of time (seconds or milliseconds) so that the interval of t within which the solution is required is not more than 0.1 second. This means that a scale factor must be chosen by which each quantity in Eq. (9) is to be multiplied in order that the resulting equation have the proper time scale, and in order that each term be expressed in volts with no term greater in magnitude than 100 volts during the solution interval. Suppose for example that we have the equation

$$dy/dx=100x^2+200, \quad 0\le x\le 2, \qquad y(x=0)=-100. \tag{13}$$

In order to achieve a suitable time scale, we introduce the variable t, expressed in milliseconds, and write

$$x=t/(50\text{ msec}), \quad 0\le t\le 100\text{ msec}. \tag{14}$$

At $x=2$, we have $y=567$, hence we introduce the variable U, measured in volts,

$$y=6U/(1\text{ volt}). \tag{15}$$

Substituting in Eq. (13), we have (omitting units),

$$50dU/dt=0.0067t^2+33, \quad 0\le t\le 100. \qquad U(t=0)=-17. \tag{16}$$

A circuit with values appropriate to solving Eq. (16) is shown in Fig. 8. For simplicity, all considerations regarding units and scale factors will be omitted in further examples.

A typical linear differential equation with constant coefficients is the harmonic oscillator equation

$$\ddot{U}+\omega^2U=0, \tag{17}$$

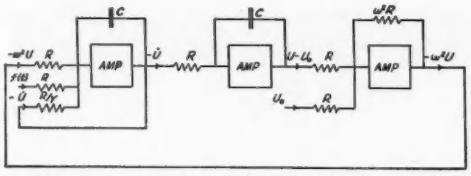


FIG. 10. Circuit for solving $\ddot{U}+\gamma\dot{U}+\omega^2U=f(t)$. ($\dot{U}_0=0$.)

where dots indicate time differentiation. Let us rewrite Eq. (17) in a form suitable for the differential analyzer:

$$\ddot{U}=-\omega^2U,$$

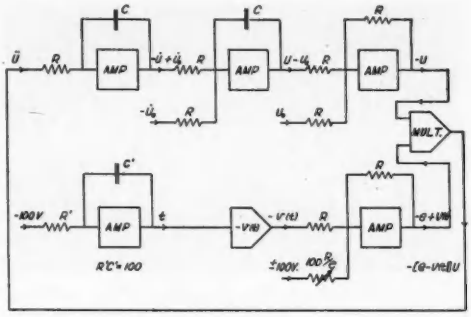
$$U=U_0+\int_0^t\dot{U}dt,$$

$$\dot{U}=\dot{U}_0+\int_0^t\ddot{U}dt. \tag{18}$$

A circuit for solving this equation is shown in Fig. 9.

The equation for a damped harmonic oscillator with forcing function $f(t)$ is

$$\ddot{U}+\gamma\dot{U}+\omega^2U=f(t). \tag{19}$$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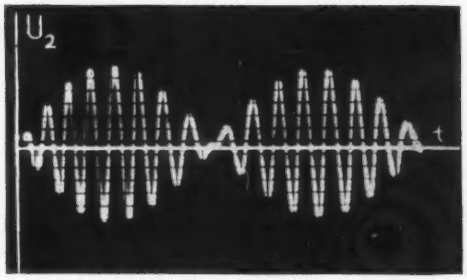
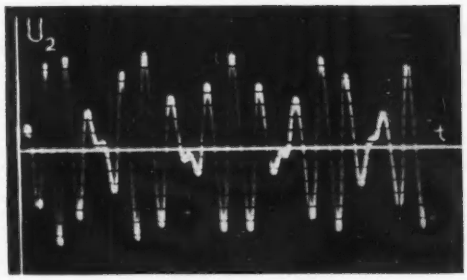
R
R



(a) Weak coupling.



(b) Strong coupling.

FIG. 13. Two coupled harmonic oscillators ($\omega_1=\omega_2$).

is the solution of a linear differential equation with constant coefficients, it may be generated by an appropriate arrangement of computing amplifiers. If not, it must be obtained from a special function generating device.

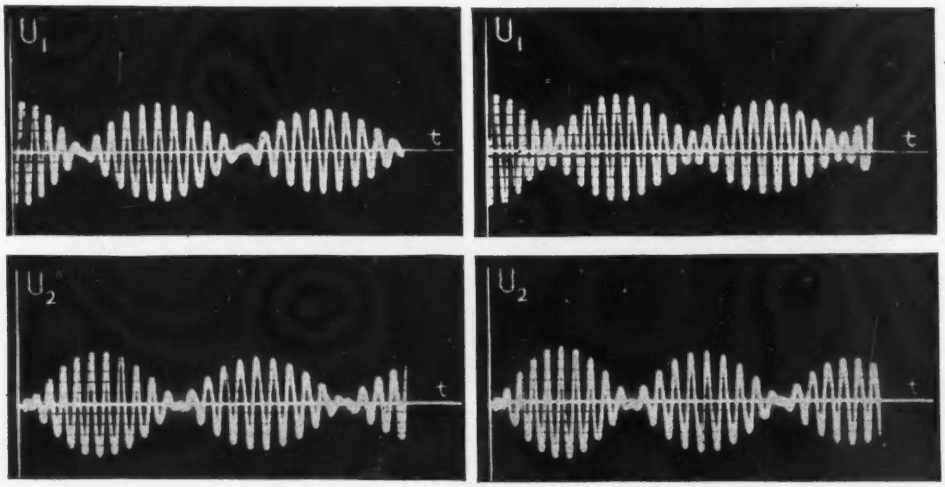
As a final example, consider the one-dimensional Schrödinger equation

$$-\frac{h^2}{2m}\frac{d^2U}{dx^2}+V(x)U=EU. \tag{21}$$

Rewriting, replacing x by t, and introducing appropriate scale factors, we can put the equation in the form

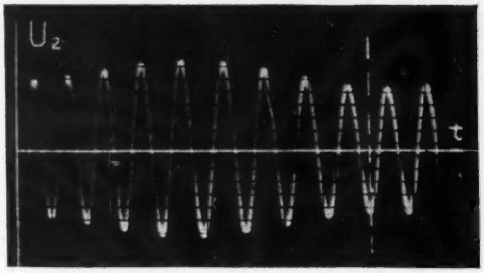
$$\ddot{U}+(e-v(t))U=0. \tag{22}$$

In order to solve this equation, we need a device to generate the function $v(t)$, and a device to multiply $(e-v(t))$ by U. Assuming such devices, we can solve this equation with the circuit of Fig. 11. To find the eigenvalues e of Eq. (22), we apply appropriate initial values U_0 and $\dot{U}_0$ to satisfy the boundary conditions at $t=0$, and vary the resistor labeled 100 R/e until the boundary condition at the other end of the range of integration is satisfied. The ratio of R to R/e then gives the eigenvalue e. Since the differential analyzer is limited inherently to equations involving only one independent variable (the time t), it cannot be used to solve partial differential equations unless they can be separated into a number of ordinary differential equations, each involving only one independent variable. The many dimensional Schrödinger equation cannot be solved directly on a differential analyzer,

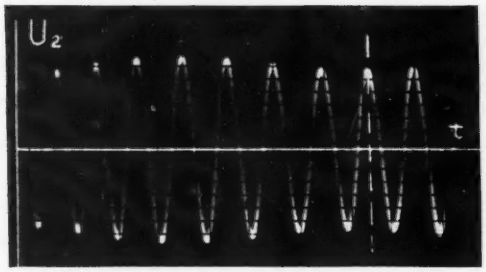


(a) $\omega_1=\omega_2$. (b) $\omega_1\neq\omega_2$.

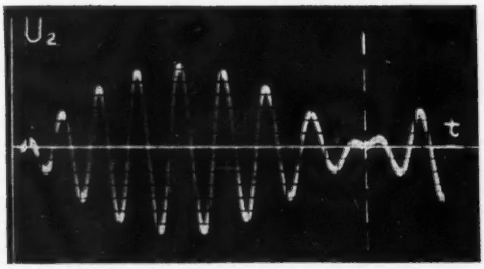
FIG. 14. Two coupled harmonic oscillators.



(a) Both oscillators excited, out of phase.



(b) Both oscillators excited, in phase.



(c) U_1 alone initially excited.

FIG. 15. Two coupled harmonic oscillators.

although the ordinary differential equations which result when the variables can be separated are solvable.

SAMPLE SOLUTIONS

Figure 12 presents a series of tracings of solutions of Eq. (19) (with $f(t)=0$) for four different values of damping, showing undamped, underdamped, critically damped, and overdamped motion.

Figure 13 shows solutions of the equations for two coupled harmonic oscillators:

$$\ddot{U}_1+\omega_1^2U_1=kU_2,$$
$$\ddot{U}_2+\omega_2^2U_2=kU_1. \tag{23}$$

Figure 13 shows $U_1(t)$ for the case $\omega_1=\omega_2$, with two different values of k, and with U_1 initially excited, U_2 at rest. Figure 14 is a study of the behavior of U_1 and U_2 when U_1 alone is initially excited. Note that when $\omega_1=\omega_2$, U_1 periodically exchanges all its energy with U_2, whereas when $\omega_1\neq\omega_2$, U_1 gives up only part of its energy to U_2.

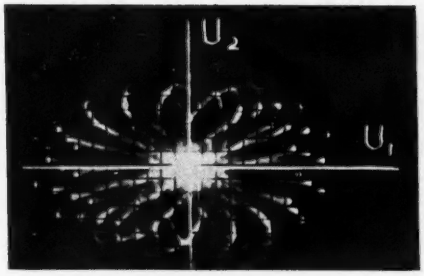


FIG. 16. Wallpaper pattern.

Figure 15 illustrates how the motion shown in Fig. 14a ($\omega_1=\omega_2$) can be regarded as a superposition of two normal modes of vibration. In the high frequency normal mode (curve (a)), U_1 and U_2 are initially excited out of phase with equal amplitudes. In the low frequency normal mode (curve (b)), U_1 and U_2 are initially excited in phase with equal amplitudes. In curve (c) of Fig. 15, U_1 alone is initially excited, and the motion of U_2 can be seen as a beat between the two normal modes of vibration, the beat frequency being the difference between the two normal mode frequencies.

Figure 16 is an example of the use of the differential analyzer as a hoot-nanny. In this case, we have the solutions shown in Fig. 14(a), with U_1 plotted horizontally and $\dot{U}_2$ vertically.

NOTES AND DISCUSSION

Shearing Stress in a Closely Coiled Helical Spring

REUBEN BENUMOF, *City College, New York, and Pace College, New York, New York*

AND

MITCHEL BENUMOV, *McKiernan-Terry Corporation, Harrison, New Jersey*

THE theory of helical springs has important applications in industry. Such springs are discussed in a number of books,[1] but unfortunately the derivation of the formulas is not indicated. The purpose of this article is to show how one of the formulas in everyday use may be obtained very simply.

This brief but accurate formula for the stress in a closely coiled helical spring, widely used by engineers, is that given by A. M. Wahl.[2] Teachers, in setting up equipment for an experiment, will find it quite valuable in that it indicates when equipment is likely to be damaged through overload. To apply the formula to a given spring, a micrometer caliper for measuring wire and coil diameters, and a slide rule for calculation are the only tools needed.

The following discussion is direct and shows how important results may be obtained by the elementary methods taught in the classroom. The formulas derived in this paper reduce to those given by Wahl for the case of a large ratio of radius of helix to radius of rod (of which the spring is composed). The method used here is elementary in the sense that the usual conditions for the equilibrium of a section are imposed. The basic assumption made is that a plane section through the spring before load is applied remains plane after the load is applied.

To simplify the analysis, it is considered that a single coil of the complete spring is isolated and two bars are attached as in Fig. 1(b). Obviously, the stresses in the isolated coil are still the same as though it were part of the complete spring. Since the coil or pitch radius is R, the moment acting on any section, such as $P-Q$, is FR, where F is the applied force.

Let two sections, say $P-Q$ and $P'-Q'$ in Fig. 1(b), be taken so as to subtend an infinitesimal central angle $d\alpha$. The length of any fiber between the sections is, then, $(R+x)d\alpha$, where x is the abscissa of the fiber.

Referring to Fig. 1(c), section $P-Q$ rotates relative to $P'-Q'$ through angle $d\phi$ about some point $(-h, k)$ under the applied moment FR. For a straight cylindrical rod, the rotation would have taken place about O. Point (x, y) in section $P-Q$ moves relative to point (x', y') in section $P'-Q'$ through distance $d\phi[(x+h)^2+(y-k)^2]^{\frac{1}{2}}$.

By definition, the modulus of elasticity in shear is

$$G=\text{stress/strain}=\tau/(d\phi/d\alpha)(1/R+x)[(x+h)^2+(y-k)^2]^{\frac{1}{2}}, \quad (1)$$

where τ is the shearing stress due to torsion. Therefore,

$$\tau=G(d\phi/d\alpha)[(x+h)^2+(y-k)^2]^{\frac{1}{2}}(R+x)^{-1}. \quad (2)$$

The section is in equilibrium. Consequently,

$$\Sigma X=0=\int\int\tau\sin\phi dydx=\int_{-r}^{r}\int_{-\sqrt{(r^2-x^2)}}^{\sqrt{(r^2-x^2)}}\tau(y-k)\times[(x+h)^2+(y-k)^2]^{-\frac{1}{2}}dydx \quad (3)$$

$$0=-2\pi k[R-(R^2-r^2)^{\frac{1}{2}}].$$

Therefore,

$$k=0. \quad (4)$$

$$\Sigma Y=0=\int\int\tau\cos\phi dydx=\int_{-r}^{r}\int_{-\sqrt{(r^2-x^2)}}^{\sqrt{(r^2-x^2)}}\tau(x+h)\times[(x+h)^2+(y-k)^2]^{-\frac{1}{2}}dydx. \quad (5)$$

Therefore,

$$h=(R/2)[1-(1-r^2/R^2)^{\frac{1}{2}}]=(R/2)\left[1-\left\{1-\left(\frac{1}{m^2}\right)\right\}^{\frac{1}{2}}\right], \quad (6)$$

where $m=R/r$. If m becomes very large, then

$$h=R/4m^2. \quad (7)$$

The load F causes a torque FR to be applied to the section.

$$FR=\int_{-r}^{r}\int_{-\sqrt{(r^2-x^2)}}^{\sqrt{(r^2-x^2)}}\tau[(x+h)^2+(y-k)^2]^{\frac{1}{2}}dydx. \quad (8)$$

Therefore,

$$(FR/G)(d\alpha/d\phi)=(\pi r^2R/2)[3-(1-r^2/R^2)^{\frac{1}{2}}]-(2\pi R^3/3)[1-(1-r^2/R^2)^{\frac{1}{2}}].$$

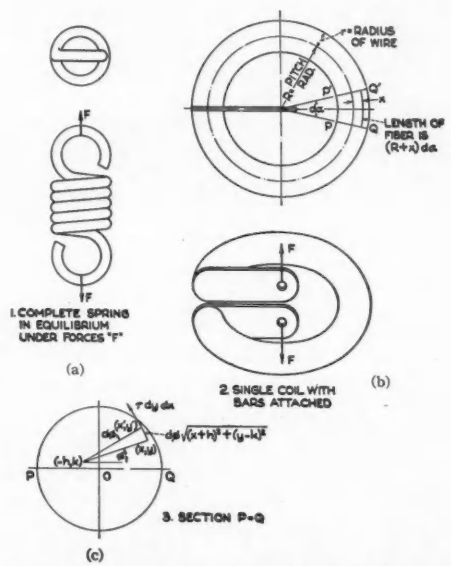


FIG. 1. Analysis of the action of a helical spring. (a) Plan and elevation views of a complete helical spring in equilibrium under forces F. (b) Plan and elevation views of a single coil in equilibrium. The short horizontal bars are imaginary, being introduced to clarify the physical situation. (c) Stresses and resultant strains in section $P-Q$ of the single coil.

As before, if $R/r=m$ becomes very large, then

$$(FR/G)(d\alpha/d\phi)=\pi r^2R/2m^2. \quad (9)$$

Stress τ is a maximum at $x=-r$. Therefore,

$$\tau_{max}=G(d\phi/d\alpha)(r-h)/(R-r)$$
$$=(F/\pi r^2)(m)(4m-1)/(2m-2). \quad (10)$$

The above formula, which is the one given by Wahl, is an expression for the torsional shearing stress. The average direct shearing stress $\tau'=F/\pi r^2$ is usually negligible in comparison to τ_{max}. However, in cases where τ' is not sufficiently small, the maximum resultant shearing stress is, to a very good approximation,

$$\tau_{max}=(F/\pi r^2)[m(4m-1)/(2m-2)+1], \quad (11)$$

where $m=R/r$.

[1] S. Timoshenko, *Strength of Materials*. Part 1. "Elementary Theory and Problems" (D. Van Nostrand Company, Inc., New York, 1930), pp. 81–84.
[2] A. M. Wahl, Trans. Am. Soc. Mech. Engrs. **51**, No. 16, 185–193, (1929).

An Entertaining Demonstration Using Window Glass as a Mirror

ALBERT V. BAEZ*
UNESCO Technical Assistance Mission, University College, Baghdad, Iraq

ALMOST every elementary physics textbook describes the following illusion: The real image of a flower appears inside a beaker full of water. Unfortunately, the aberrations inherent in the large concave mirror which is required make the illusion visible only in a very narrow cone of space.

The following demonstration has the same element of surprise and can be performed with an ordinary piece of window glass. Window glass is easily obtainable and inexpensive. A plane mirror is, in principle, free of aberrations, and the demonstration can be set up to be clearly visible to an entire seated audience.

The principles are as old as optics. The new features of the following information center around the entertainment and instructional potential of the demonstration.

In Fig. 1, a socket A_1 with a frosted bulb B_1 is placed in front of a large piece of window glass W. The bulb and socket are hidden from the class by a cover C, which is painted black inside. The virtual image of A_1, B_1 formed by the glass acting as a reflector is at A_2', B_2'. A real socket A_2 and a real bulb B_2 are placed to coincide with the image A_2', B_2'.

The wires which go to the real socket at A_2 are shorted there, although from a distance the student thinks that they are connected to B_2 and controlled by a switch S. Actually S controls the current to bulb B_1 only. A footswitch F, which the class cannot see, is connected as shown. After connecting plug P to the house current and starting with S open and F closed, the demonstration proceeds as follows:

1. The students can view A_2, B_2 directly through the glass.

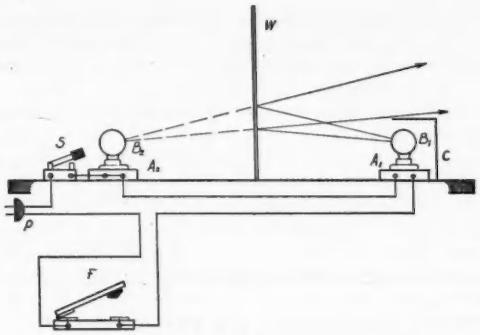


FIG. 1. Identical frosted bulbs B_1 and B_2 are placed in identical sockets A_1 and A_2, respectively, and situated symmetrically with respect to the window glass W. When B_1 lights, B_2 appears lit.

2. By closing S, the bulb B_1 lights up and its image B_2' makes B_2 appear lit. The students think that S has turned on B_2.

3. By unscrewing B_2 and opening F it seems that B_2 has gone out because it was unscrewed. B_2 is put aside and no bulb appears in A_2. This occurs because in ordinary classroom illumination the image of B_1 is visible only when B_1 is lit.

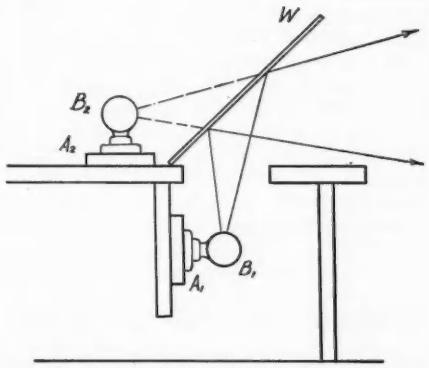


FIG. 2. This arrangement of the optical parts is possible when the lecture table has a hole in it. The electrical connections are identical with those of Fig. 1.

4. Next, F is closed, whereupon B_1 lights up producing a lit image at B_2'. To the students a *lit* bulb seems to have appeared in A_2. Time out for surprise.

5. But just as suddenly, by opening F, it disappears.

6. Apparently confused, the instructor puts B_2 back into A_2. When he finishes screwing it in, it lights! (Or seems to, when F is closed, of course.)

7. In disgust, the instructor takes a hammer and strikes B_2 a blow which smashes it. Yet to the students who see a lit bulb at B_2', the bulb B_2 seems intact and lit! (Obviously, B_2 can be a bulb with a burned out filament, and A_2 must be held firmly in its place).

This finishes the entertainment portion of the demonstration.

The instructional aspects concerning geometrical optics, transmission and reflection, virtual images, parallax, and the quality of the image formed by a plane mirror are left to the instructor.

Figure 2 shows an arrangement suggested by Professor Paul Kirkpatrick for use with a lecture demonstration table with a hole in it. The arrangement in Fig. 1 has been found quite suitable in a lecture hall with a sloping floor.

* On leave from the University of Redlands, Redlands, California.

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LETTERS TO THE EDITOR

The Behavior of a Flat Whisky Bottle When Heated

IN earlier notes[1,2] the *mechanical* effects of pressure upon a flat whisky bottle were reported. A conventional whisky bottle is equipped with a tight-fitting, one-hole stopper into which is inserted a long glass tube of small bore. The bottle is filled with colored water; the initial level of the water is above the stopper in the tube. When the *wide* faces are squeezed by hand the liquid level rises; when the edges are squeezed the liquid level falls. Professor Dodd accounts for this nicely in terms of changing cross-sectional area.

This note reports the very interesting scheme of things when the flask is *heated*. If a *wide* face is heated locally by directing a Bunsen flame upon it, the liquid level falls. This is to be expected, since the face expands, the bottle "rounds" up, and its volume is increased.

Now if an *edge* is heated how will things go? The result quite astonished me.

JULIUS SUMNER MILLER

Dillard University,
New Orleans 22, Louisiana

[1] J. S. Miller, Am. J. Phys. **18**, 164 (1950).
[2] L. E. Dodd, Am. J. Phys. **18**, 398 (1950).

Kundt's Tube Projection Demonstration

THE usual Kundt's tube, consisting of a cylindrical glass tube, was found unsatisfactory for classroom projection purposes because the refractions introduced by the curved glass surfaces rendered projection impossible.

A rectangular box measuring about 2 in.×2 in.×30 in. was made of wood except that two of the long faces opposite one another were made of flat glass. At one of the ends an ordinary earphone was attached permanently. Before closing the box a transparent celluloid ruler was cemented to one of the glass sides and some fine cork dust was introduced. The box was closed and screwed tight to make it airtight.

The earphone was connected directly to the output of a variable audiofrequency generator and resonances were observed at several values of the exciting frequency. A standard-size lantern-slide projector, in which the bellows connecting the lantern-slide holder support and projection lens were easily removable, was used. By removing the bellows from the lantern-slide holder support it was found that the rectangular box could be rested on the long rods which supported the projection lens holder. The lens could then be moved out so that the celluloid scale and the cork dust simultaneously appeared focused in the image on the screen.

The curious behavior of the cork dust particles as projected on the screen could be clearly seen by a large class. The enlarged image of the celluloid scale made a quantitative measure of the distance between nodes possible. This, together with the frequency reading on the oscillator, made possible a determination of the speed of sound in class. Since the long axis of the box was normal to the optic axis of the projection lens, it was possible to slide the box along its length and observe the pattern of the excited cork dust at different places within the box.

As an aside it may be mentioned that during the same lecture a loud-speaker placed near a soap film whose image was projected on a screen produced interesting patterns at certain frequencies (eigenvalues) impressed on the loud-speaker by the audiofrequency oscillator. The fact that the films produce colors by interference makes them more attractive, although the colors are irrelevant to the sound vibration demonstration.

ALBERT V. BAEZ*

UNESCO Technical Assistance Mission,
University College, Baghdad, Iraq

* On leave from the University of Redlands, Redlands, California.

Remarks Concerning the Paper "Conditions for the Derivation of the Stress Deviator Tensor"

I BELIEVE that it may be appropriate to make the following comments (largely relating to terminology) concerning O. L. Anderson's interesting paper[1] of the above title.

The author seeks the "physical requirements, i.e., *conditions*, of such a decomposition" (of the stress tensor into a deviator and a spherical component). Now this cannot mean the physical conditions which must be satisfied in order that such a decomposition be possible, for these conditions are neither more nor less than the conditions which make it possible to represent stress by a a second-order symmetric tensor. For *any* physical quantity which is representable as a second-order symmetric tensor, e.g., the inertia tensor of a rigid body in dynamics, such

decomposition is possible. It appears, therefore, that the author is concerned not with the *possibility* of such decomposition but rather with its *usefulness*, and in particular, its usefulness in the theory of plasticity. In this connection we remark that such decomposition had previously been used in other disciplines, such as elasticity theory[2] and hydrodynamics,[3] and was carried through in substantially the same way as described by the author, which he appears to attribute to Russian writers on plasticity.

Since we are concerned with fundamental physical meaning, it may be well to mention that the yield criteria listed are those for a perfectly plastic material, which some real materials approximate; they are false for work-hardening materials,[4] although the criterion can be expressed in terms of the invariants of the stress deviator; they are false for metal single crystals,[5] and the criterion for yield is not expressible in terms of the invariants of the stress deviator in this case.

In the third paragraph the author says: "The question arises, are there other ways of decomposing S (the stress tensor) so that one of its constituents produces the desired yield criteria? If this decomposition is unique, what physical meaning is implied by this uniqueness itself? What is implied, in terms of elastic theory, by making this decomposition unique?" I am at a loss as to the meaning of the last statement. It seems to me that the decomposition either is, or is not, unique in the sense given by the first sentence. How can we *make* it unique?

We note that the results in the second section of the paper may be deduced[6] from (1) definition of stress, (2) definition of (infinitesimal) strain, (3) symmetry of the stress tensor, (4) generalized Hooke's law, and (5) elastic isotropy. In particular, if item (5) is dropped, the results are quite different, there being in the least symmetric case 21 independent elastic constants instead of two.

In the third section the author assumes (in addition to the implicit assumption of the five items just mentioned) "two postulatory conditions: (1) the stress and strain tensors can be decomposed into two and only two fundamentally different independent constituents, and (2) the total elastic strain energy is simply the sum of the strain energy of the individual constituents." It would have been sufficient for the succeeding deductions to have assumed that the stress and strain tensors can be decomposed into two constituents such that the total strain energy is the sum of the strain energies associated with the two constituents taken separately. In particular, the author's characterization of "fundamentally independent" appears to be irrelevant to the argument which he presents.

In the final section of the paper the author states: "If a strain is proportional to a stress, i.e., if $\epsilon_{ik}=a\sigma_{ik}$, Hooke's law is valid for all components of the stress and strain. It is well known that this proportionality is not observed in the general strain-stress relationships as shown by Eq. (12) which indicates that each of the ϵ_{kk} terms is a linear function of all the σ_{kk} terms. The lack of a causal relationship is more forcibly demonstrated for the case of simple axial extension in which there is lateral strain with no component of stress in the direction of that strain. The question arises, what causes the lateral strain? The concept of Poisson's ratio cannot explain away the above, for this number is experimentally determined and has no theoretical significance." The set of equations $\sigma_{ij}=C_{ijkl}\epsilon_{kl}$ expressing the stress components as linear functions of the strain components is known as the generalized Hooke's law[6] and is usually regarded as expressing a causal connection between stress and strain. The author's implied restriction on the concept of causality is an innovation of doubtful value, since it would deny causality in numerous situations where a functional relationship exists among three or more physical entities, while affirming causality where a functional relation involves only two physical entities. The answer to the question raised by the author is, in my opinion, that the lateral strain is caused by the longitudinal stress. There is nothing peculiar about this behavior if it is viewed in the light of the general anisotropic case, where a change in a single stress component produces an alteration of every strain component. That the Poisson ratio is determined by experiment is not unusual; all elastic constants are thus determined. I do not see that the Poisson ratio has any more or any less theoretical significance than any other elastic constant.

In conclusion, the physical significance which the author attaches to the stress deviator and the spherical stress tensor cannot be of general validity. If the number of independent elastic constants exceeds two, the argument must necessarily fail. Perhaps in the general anisotropic case it may be possible to decompose the stress tensor S and the strain tensor E into 21 components, $S=\sum_{i=1}^{21} S_i$ and $E=\sum_{i=1}^{21} E_i$, such that $E_i=a_iS_i(i=1, 2, \cdots, 21)$. However, even if this could be done, and if the decomposition could be proved to be unique, I should still be unwilling to concede that the S_i enjoyed any special physical significance not enjoyed by any other set of constituents. Such a resolution might be *useful*, but I do not see that it would be *physically more meaningful*.

MILTON O. PEACH

University of Notre Dame,
Notre Dame, Indiana

[1] Am. J. Phys. **20**, 236 (1952).
[2] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Cambridge University Press, Cambridge, 1927), fourth edition, p. 83.
[3] H. Lamb, *Hydrodynamics* (Cambridge University Press, Cambridge, 1932), sixth edition, p. 574.
[4] A. M. Freudenthal, *The Inelastic Behavior of Engineering Materials and Structures* (John Wiley and Sons, Inc., New York, 1950), p. 301.
[5] See reference 4, page 129.
[6] I. S. Sokolnikoff, *Mathematical Theory of Elasticity* (McGraw-Hill Book Company, Inc., New York, 1946), Chap. III.

Extension of Remarks on the Derivation of the Stress Deviator Tensor

THE remarks[1] of Mr. Peach concerning my paper,[2] "The Derivation of the Stress Deviator Tensor," deserve further explanation.

In his second paragraph, Mr. Peach felt that the paper had to be concerned with either the *possibility* or *usefulness* of the decomposition of the stress and strain tensor, implying that there were no other reasons for the paper.

Now the paper was not intended to show that the decomposition was *possible*, for it should be evident that any tensor quantity, symmetric or otherwise, can be decomposed in an infinite number of ways. The paper was not intended to show that the decomposition was *useful*, a point I emphasized in the first paragraph of my paper. References 1, 2, 3, and 4 of my paper were listed to give ample evidence of usefulness of this decomposition. The paper was intended to show that the particular decomposition was *significant*, a possibility Mr. Peach apparently overlooked.

As Mr. Peach says, "this decomposition has previously been used in elasticity theory and hydrodynamics," but the fact that the decomposition is well known is irrelevant. The point of the paper is (a) workers in plasticity have found this decomposition particularly useful, (b) this decomposition is significant because it requires that the strain energy have a simplified form, and (c) this form implies that the constituents of stress and strain are independent of each other, a fact that has not been noted previous to this paper.

In Mr. Peach's third paragraph, he states that the yield criteria are "false for work-hardened materials . . . and false for metal single crystals." This should not be so surprising, since these materials are excluded by the assumptions made in the derivations of the yield criteria. In work-hardened materials, there is an internal distribution of strain which is unrelated to the external stress; hence, there is no well-defined strain-energy relation. For metal single crystals, there are more than two independent elastic constants, and the strain-energy expression has a form different from that used in the derivations in my paper.

In Mr. Peach's sixth paragraph, he makes the following statement, "It would have been sufficient for the succeeding deductions to have assumed that the stress and strain tensors can be decomposed into two constituents such that the total strain energy is the sum of the strain energies associated with the two constituents taken separately. In particular, the author's characterization of 'fundamentally independent' appears to be irrelevant to the argument he presents." Mr. Peach erroneously confuses an equivalent argument with an irrelevant argument. Let us follow Mr. Peach's line of reasoning.

Let $\mathbf{A}=\mathbf{A}_1+\mathbf{A}_2$ and $\mathbf{B}=\mathbf{B}_1+\mathbf{B}_2$; then let us assume that $\mathbf{A}\cdot\mathbf{B}=\mathbf{A}_1\cdot\mathbf{B}_1+\mathbf{A}_2\cdot\mathbf{B}_2$ for all values of $\mathbf{A}$ and $\mathbf{B}$.

This means that $\mathbf{A}_1$ acting on $\mathbf{B}_2$ plus $\mathbf{A}_2$ acting on $\mathbf{B}_1$ is null. Now if $\mathbf{A}_1$ were not independent of $\mathbf{B}_2$, and likewise if $\mathbf{A}_2$ were not independent of $\mathbf{B}_1$, we could take arbitrary values of $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$, $\mathbf{B}_2$ such that the sum

$$\mathbf{A}_1\cdot\mathbf{B}_2+\mathbf{A}_2\cdot\mathbf{B}_1,$$

would be different from zero, contrary to our assumption. I cannot agree that the characterization of "fundamentally independent" is irrelevant to the energy argument. It is rather equivalent.

In Mr. Peach's seventh paragraph, he objects to my restriction on causality. The concept of causality is a philosophical question and is used variously by different authorities. Let us avoid the work and discuss the meaning of Hooke's law itself with regard to lateral strains. In G. Joos' *Treatise on Theoretical Physics*, we find: "the foundation of the so-called classical theory of elasticity is Hooke's law, which states that the *normal* stress is proportional to the elongation per unit length (*ut tensio, sic vis*)." The Hookian relation does not include lateral strains. While the generalized Hooke's law is a useful functional relationship, it is not a fundamental one since there is no principle of mechanics which provides for orthogonal responses to forces. The cause of a lateral strain can only indirectly be the longitudinal force. The direct cause must be a stress component in the direction of the lateral strain—otherwise Hooke's law cannot explain the lateral strain. That such a stress component exists is shown by my paper. Consider an elemental domain under a simple tensile load along the x axis. According to the theory, we have for the dilatational stress system,

$$\sigma_x''=\sigma_y''=\sigma_z''=\tfrac{1}{3}P,$$

and for the distorting stress system,

$$\sigma_x'=\tfrac{2}{3}P,\ \sigma_y'=-\tfrac{1}{3}P,\ \sigma_z'=-\tfrac{1}{3}P.$$

These two stress systems are equivalent to a tensile load, since $\sigma_x'+\sigma_x''=P$, $\sigma_y'+\sigma_y''=0$, and $\sigma_z'+\sigma_z''=0$. Yet, they predict a net lateral strain since the elastic modulus for the distorting and dilatational stress systems are different. In this example, there is no reliance on a generalized Hooke's law. Hooke's law is used, *as he gave it*, to produce a lateral extension. It is necessary to assume, however, that there are two fundamentally different types of stress systems. To argue that the tensile load produces the lateral strain misses the point. Unless one relies on a complicated argument based on directed valence forces between atoms (of which we know too little) a normal force cannot be considered as the prime cause of a lateral strain and be consistent with the principles of Newtonian mechanics.

In the seventh paragraph, Mr. Peach states, "I do not see that the Poisson ratio has any more or any less theoretical significance than any other elastic constant." While the magnitude of the elastic constants are all determined by experiment, there is a difference in our knowledge concerning the derivation of these quantities. The bulk modulus is defined by a precise thermodynamic equation and can be calculated provided one knows the equation of state. The other elastic moduli (excluding Poisson's ratio) are all related to the second derivatives of the potential of the chemical binding energies of the atoms themselves. While the evaluation of these elastic moduli is difficult in practice, they are nevertheless well known in principle and are as fundamental as our knowledge of atomic forces themselves. Poisson's ratio is named after a French mathematician, who determined it using a postulated molecular theory of structure. Poisson's model does not account for chemical binding energies in the form known to us today, however, and we cannot consider it as having theoretical significance. All we can say for the Poisson ratio today is that it is a ratio of certain elastic moduli. I submit that the elastic moduli, excepting Poisson's ratio, have a fundamental theoretical significance.

To dismiss the previous arguments on the premise that they are only valid for isotropic materials (as Mr. Peach would do in his last paragraph) is contrary to the spirit of

the development of the science of elasticity. I dare say over 95 percent of all elasticity papers are valid only for isotropic materials. Airy's function itself is derived with the assumption of isotropy. Our knowledge of the mechanics of crystal systems is meager compared to our knowledge of the mechanics of isotropic systems. This does not prevent us from making progress, restrictive as it may be, when we can.

While Mr. Peach does not agree that the decomposition of the tensors made in my paper is unique, he fails to offer another set. I should be very interested to be shown another set of constituents subjected to the conditions of Eqs. (11), (12), and (13) of my paper.

There is yet one minor point of contention. If Mr. Peach will read the paper carefully, he will note that I do not attribute the original decomposition of stress to Russian authors of plasticity; I only give their method by way of a review of the problem.

ORSON L. ANDERSON*

University of Utah,
Salt Lake City, Utah

[1] Milton O. Peach, Am. J. Phys. **21**, 66 (1953).
[2] O. L. Anderson, Am. J. Phys. **20**, 236 (1952).
* Presently a member of the Technical Staff, Bell Telephone Laboratories, Murray Hill, New Jersey.

Heat Transfer to a Boiling Liquid

THE letter by Mr. Edwards[1] on the phenomena observed when a warm body is immersed in liquid air recalled the following measurements made several years ago on the same phenomena.

When a warm body is placed in liquid air, boiling occurs at the warm surface. Vaporized liquid air is evolved at a moderate and steady rate until just before the body reaches the temperature of the liquid air. Then occurs very violent boiling that quickly subsides as the body attains liquid air temperature.

This phenomenon is not a property of liquid air, but of boiling liquids. The same series of events occurs when a hot body is dropped into boiling water. The two types of heat transfer, the efficient "contact" transfer and the less efficient transfer through a gas film, are well known to heat engineers,[2] but not the mechanism suggested below for the former.

A conical piece of copper was set base-upward into a sleeve of nickle alloy tube so that only a circular surface of copper was exposed. The sleeve was an alloy with low thermal conductivity. An electrical coil was wound on the cone. A constantan wire was soldered at the base of a small hole that reached almost to the exposed surface of the copper cone from inside the sleeve. This formed a thermocouple junction that was connected to measure the temperature difference between the copper surface, heated by power metered to the coil, and surrounding liquid nitrogen when the apparatus was immersed therein. The sleeve kept liquid nitrogen off the unexposed parts of the cone.

Stable temperature-power conditions were obtained over the range shown in Fig. 1 for "contact" transfer. The points on the two curves were not all obtained in sequence, as more and more power was applied; but the power successively increased and decreased, each time retracing the same curve. For power inputs over 72 watts/cm² the "contact" type of transfer ceased and the temperature of the surface rose several hundred degrees. The lower curve with the reciprocal slope of 2.8 watts/cm² deg was obtained when a thin film of ice coated the copper surface.

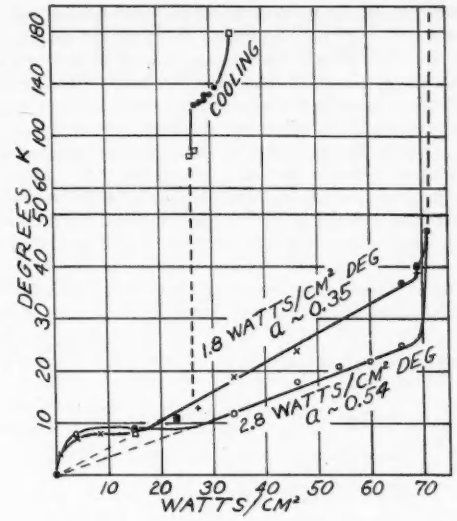

FIG. 1. Temperature difference between liquid nitrogen and solid surface vs rate of heat transfer. The two lower curves represent "contact" or thin-film transfer, the upper for a copper surface, the lower for a copper surface sheathed in ice. The upper curve represents heat transfer through a thick film of gas.

The "cooling" curve represents heat transfer through a gas film. The surface temperatures represented by points on this curve were not as stable as those for contact cooling, but always the "contact" type of transfer took over when the power was reduced below 26 watts/cm².

The word "contact" transfer suggests the unsatisfactory picture of the liquid wetting a surface whose temperature exceeds the boiling point of the liquid. It is here proposed that so-called "contact" transfer is really transfer through a film of gas whose thickness is less than the mean free path of molecules in the gas. Into this gas layer molecules evaporate from the liquid surface and strike the hot solid surface without intermediate collisions. There they take up energy and return to the liquid.

The theory of this type of heat transfer[3] shows the heat transfer independent of the thickness of the gas film, roughly proportional to the pressure and the temperature difference between the bounding surfaces, and dependent upon "accommodation coefficients" for the gas in contact with each surface. Accommodation coefficients are one or less and represent the effectiveness of the surface in imparting energy to the molecules.

If the accommodation coefficient for the liquid nitrogen surface is assumed to be one (it may be less), then a not unreasonable accommodation coefficient for the copper surface of about $a=0.35$ will account for the upper curve and $a=0.54$ for the ice-coated surface of the lower curve.

These values assume that the whole surface is conducting and no attached bubbles reduce the effective area.

The "noncontact" heat transfer is effected through a film of gas much thicker than the mean free path. If ordinary conduction theory is applied to the last stable point on the "cooling" curve and the observed data used to calculate the average film thickness, one finds that the thickness is about 10 mean free paths. No account of temperature jumps[4] at the surfaces entered this estimate. With jumps included the average thickness would be less.

It is easily conceivable that the switch from gas film heat transfer to "contact," or more appropriately, thin-film transfer, occurs when a reasonable fraction of the surface is covered by a film only a few mean free paths thick.

J. Elmer Rhodes, Jr.

Georgia Institute of Technology, Atlanta, Georgia

[1] M. H. Edwards, Am. J. Phys. **20**, 313 (1952).
[2] M. Jakob, *Heat Transfer* (John Wiley and Sons, Inc., New York, (1949), Vol. I, Chapter 29.
[3] E. H. Kennard, *Kinetic Theory of Gases* (McGraw-Hill Book Company, Inc., New York, 1938), p. 316.
[4] See reference 3, p. 312.

Frequencies Resulting from Distortion

In a recent Letter to the Editor[1] Howard N. Maxwell asks for a physical explanation of the additional frequencies found by a Fourier analysis of a distorted monochromatic wave train. It is evident that such frequencies are not due to resonances in the distorting device itself, whether that is an electrical rectifier or a nonlinear mechanical device. Where then do the additional frequencies arise?

This question reminds me of the hot controversy concerning the real existence of sidebands in amplitude-modulated radio transmission, fought out in the popular radio journals about twenty years ago. If a carrier wave of frequency F is modulated at a much lower frequency f, a suitable wavemeter will show responses at the sideband frequencies $F \pm f$; yet there is no circuit in the transmitter tuned to either of these sideband frequencies. The conclusion then and now accepted is that we have two mathematical ways of representing the phenomenon, both equally valid; namely, an equation showing it as a modulated wave, and another showing it as the carrier plus two sidebands.

When Helmholtz analyzed the wave forms set up by musical instruments, he obtained harmonic responses in his cavity resonators; and he indicated the modes of vibration of strings and air-columns that would account for the presence of the harmonics in the wave form. Perhaps this is why we instinctively look for similar mechanisms when we observe a distorted wave of any kind. However, what is appropriate in acoustics is not necessarily so in other branches of physics. If Dr. Maxwell is asking whether we can identify an original source of each harmonic in a distorted wave, no matter what may be the cause of the distortion, I suggest that a wish to do so may be a survival from the too-mechanistic point of view of the nineteenth century.

Chas. Williamson

Carnegie Institute of Technology, Pittsburgh 13, Pennsylvania

[1] Howard N. Maxwell, Am. J. Phys. **20**, 310 (1952).

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ANNOUNCEMENTS AND NEWS

Book Reviews

Industrial High Vacuum. J. R. Davy. Pp. 243+viii, Figs. 82, 14.5×22 cm. Sir Isaac Pitman & Sons, Ltd., London. Price 25s.

If high vacuum is a technology which sometimes rises to the dignity of a science, it is also an art and a cult. Devotees chat and reminisce, and compare notes and recipes in the language of their cult, safe in the knowledge that they will be understood by their brethren. They even write books—indeed many of this kind have been written in the last few years. These volumes must be distinguished sharply from the masterworks of the giants—Dushman, Dunnoyer, Espé and Knoll, Strong—yet they have individual usefulness as each devotee describes the rituals in which he has participated.

The volume under consideration deals expertly, and in more than half its pages, with the production and resulting properties of thin films of metals, metalloids, and salts; the manufacture of mirrors, antireflection coatings, and selective transmission filters. Just how to set about these tasks is detailed from the ground up—the selection and fitting of the room, the components of the apparatus, choice of operating chamber with its demountable leads, control knobs, etc. The source-furnace and raw materials for evaporation are given in variety by one who is thoroughly familiar with them. Particularly good are the sectioned sketches of gaskets, rotary seals, and other closure devices; and the directions for discovering and curing leaks.

The basic or peripheral information, which would be required by those not yet of the cult, is scanty in the extreme. There is one picture of a condensation pump, of a type little known in the United States. The fractionating pump, the choice and properties of pump oils, and the art of making and maintaining a high vacuum are virtually absent. The chapter on measuring vacuum is equally sketchy—a cultist would, of course, know where to look for information! In this vein, the section on kinetic theory, which occupies the first four voluminous chapters of Dushman, is relegated to the last eight pages of the present book.

A relatively comprehensive chapter on Molecular Distillation is included, presumably because of the early important contributions made in the field by British scientists and manufacturers. The story has been compiled from scientific articles and the patent literature but is, on the whole, well told.

If, then, we rate this book on what it treats, it is a good book. If we rate it on the contents implied by its title, it is less admirable. But for anyone requiring a treatise on "The Art and Practice of Surface Film Deposition in Vacuo," the volume will be indispensable. Also, it provides the American operator with a first-hand description of British methods.

KENNETH HICKMAN
Rochester, New York

The Astronomical Universe. WASLEY S. KROGDAHL. Pp. 599+xvii, Figs. 275, 16×24 cm. The Macmillan Company, New York, 1952. Price $6.25.

Professor Krogdahl offers a distinctively new and different approach in the field of astronomy to the textbook problems created by the changed emphases in preparatory education and the newer evaluations of the objectives of college science courses for nonmajors.

This is not a watered-down conventional text with frustrating inadequacy of explanations, nor a purely descriptive book, but a serious and skillful effort to meet both the proper objectives of a science course and the abilities of the typical liberal arts student as well as to provide a solid, stimulating account of astronomy. The preface indicates the author's belief in its suitability for astronomy majors also. For the latter purpose it suffers serious inadequacies. On its effectiveness for the primary purpose, there will probably be as many divergent opinions as there are teachers of astronomy; only experience with classes will give an answer. The subsequent remarks are merely one teacher's preliminary reactions. Certainly the text merits serious examination by all teachers of astronomy for general students.

The book's distinctions are a new approach to the order of topics and to the method of discussion and explanation, and unusual skill in exposition and vitality of expression. Specific, articulate emphasis is placed on the scientific method, which is discussed in the introduction in a general way. Section I opens unconventionally with a complete account of Kepler's and Newton's laws and gravitation, in which the operation of the scientific method is brought out descriptively but with little analysis of the specific reasoning. Other unusual orderings of topics appear throughout the book for the same reason.

The historical order is ignored. Names and dates are almost completely omitted as superfluous burdens to the student. The order of discoveries is not brought out. While these omissions may make possible more internally consistent demonstrations of the scientific method, they involve a loss of perspective of the science and leave one with a curious sense of lack of moorings. A more historical order, developing topics from general, historically presented observations, would deepen the student's appreciation of the nature of science and need not interfere with, but would often enhance, his comprehension of the scientific method.

The subject is developed with rigor and thoroughness in spite of avoidance of algebraic formulations. This avoidance is skillfully and successfully accomplished without loss of quantitative reasoning in the treatment of the solar system; gravitational problems are presented clearly in successive steps without the use of complete formulas. In physical problems, there is less success in this; radiation laws are used in algebraic form where a similar approach would be more real to average students.

The writing is exceptionally clear, efficient, lively, and characterized by an admirably light touch. There is a tendency to lose some clarity, occasionally, by use of what most students would call (deplorably) "big words." In discussions of many topics in modern astrophysics ambition for completeness and thoroughness has outstripped the typical student's digestive ability for brief, almost purely verbal explanation of complicated concepts. His ability and need to comprehend the topics in such full detail is highly questionable.

The book is very up to date. Brief accounts, rather cursory, are given of such topics as radio astronomy, shell stars, and recent thinking in cosmogony. The speed of advance in the science has caused a few inconsistencies; the account of Jupiter's satellites includes the twelfth, but "Jupiter's outer three" are referred to in the discussion of Saturn's satellites, for example.

Illustrations are well chosen and, in the main, well reproduced. Line drawings are meticulous and original, though not always distinguished by ready clarity, and often lacking where needed. Extensive and comprehensive question and problem lists accompany the chapters, many of them quite original and thought provoking, and some of them real teasers. Star charts (transverse Mercator projections), with white stars on blue background, are well drawn with minimum distortion, but suffer from small scale and inclusion of too many faint stars, looking too bright.

PHILIP S. RIGGS
Drake University

Ultrasonic Physics. E. G. RICHARDSON. Pp. 285+x, Figs. 100, 15.5×23.5 cm. Elsevier Publishing Company, Houston, Texas, 1952. Price $5.00.

The field of ultrasonics has expanded during the past quarter century to a point where it does not seem possible to cover the entire subject comprehensively in any single volume. E. G. Richardson has wisely chosen to restrict his new book to those phases concerning the propagation of ultrasonic waves in relation to molecular physics. In several instances, associated material involving ultrasonic waves has also been included.

Prime emphasis is placed on the construction and use of acoustical interferometers. After introductory chapters on the production and detection of ultrasonic waves, the author describes experimental techniques for the determination of velocity and absorption in gases, liquids, and solids. Only sufficient theory is given to show generally the relation of these parameters to molecular structure. One

chapter is devoted to summarizing very briefly the theories concerning anomalous velocity and absorption in various media. The final chapter describes the effects associated with the propagation of ultrasonic waves through dispersed systems. A bibliography follows each of the eight chapters.

In this book the author is at his best in the discussion of research with the acoustical interferometer, particularly those aspects with which he has had personal experience. In several cases, however, he seems to have sacrificed clarity for brevity to an undesirable extent. To cite a few examples, the discussion of cavitation (page 56) not only is inadequate but is actually misleading in terms of generally accepted explanations of cavitation. In the treatment of the effect of electrolytes on ultrasonic absorption, data are presented without any statement as to the frequency. The author uses in excess of three pages to discuss the Debye effect involving ionic vibration potentials and yet fails to indicate that the effect has been experimentally verified for simple ionic solutions (as of 1949). It is regrettable that there is not a discussion of the spherical resonator technique used by Tamm in Germany and Leonard and his co-workers in this country for the determination of ultrasonic absorption. This method seems to be the only one capable of yielding results for liquids of moderate absorption at frequencies from 30 to 500 kc/sec at the present time.

In general, the organization of this book is adequate. With the possible exception of the first chapter, the material is presented in such a fashion as to constitute relatively "easy reading."

This book should prove interesting to the person who has relatively little experience with ultrasonic techniques and would like to learn of their potentialities as a tool for the study of molecular structure. It should be within the grasp of any advanced undergraduate student in physics and physical chemistry.

EARNEST YEAGER
Western Reserve University

Elements of Wave Mechanics. N. F. MOTT. Pp. 156+ix, Figs. 30, 14.5×22 cm. Cambridge University Press, New York. Price $3.75.

To those acquainted with the author's *Outline of Wave Mechanics*, published in 1930 and now out of print, Professor Mott's *Elements of Wave Mechanics* will come as a welcome addition to the literature. The book, like its predecessor, is intended for students who would presumably correspond, in this country, to undergraduates in their senior year. Assuming that the training of experimental physicists at this level should differ from that of students of theoretical physics, the book is aimed at the former group. It is, of course, to be regarded as an introduction to more advanced treatises, as the author is careful to point out. With this in mind, it is understandable that the book is very brief and that many subjects must be treated in an almost cursory fashion.

As in the *Outline* the approach is confined to the wave treatment, and no mention can be made of the more powerful methods and more general point of view which become available in a more advanced treatment. Continuing the comparison from the point of view of subject matter, the present book not only brings the discussion up to date but it covers a somewhat wider range of topics than does its predecessor. Beginning with a mathematical introduction, which includes a brief discussion of differential equations and Fourier's theorem, the author proceeds to the quantum-mechanical description via the Schroedinger equation. After a very much simplified discussion of the wave properties of matter and of the uncertainty principle, a more detailed account of stationary state and dynamical processes follows. As is to be expected in a book with the avowed purpose of serving as a brief introduction to the subject, the choice of topics must be a conventional one. For example, under the heading Stationary States, one will find discussions of the hydrogen atom, harmonic oscillator, perturbation and variational methods, effects of electric and magnetic fields, and electron spin. Of course, the treatment of these various problems in some cases may serve only to whet the appetite, and this may very well be all to the good. The many-body problems also receive attention, and here a simplified version of the problems of molecular structure, chemical binding, and the solid state is presented in some 37 pages. The chapter on quantum dynamics has been extended and amplified considerably as compared to the *Outline*. The book closes with brief, largely qualitative comments concerning relativistic wave mechanics and nuclear physics (alpha- and beta-decay, in particular). Throughout the text a number of appropriate problems have been interspersed.

There are a number of pedagogical devices which appear in the book and which were particularly pleasing to this reviewer. For instance, the early introduction of the WKB solution of the wave equation for the purpose of providing a (qualitative) understanding of how a wave function behaves is to be applauded. On the debit side the excessive use of reasoning by analogy is perhaps subject to adverse criticism. Thus, the wave equation is introduced by the conjunction of the Davisson-Germer wavelength–momentum relation and the scalar wave equation describing sound waves (or superficially, electromagnetic waves in vacuum). This, it seems to this reviewer, confuses the issue since DeBroglie waves are like neither of these classical waves physically. This is exemplified by the well-known fact that the procedure adopted does not lead to the correct time-dependence in a natural way. It seems that, without violating the requirements of brevity and simplicity, the less objectionable procedure of utilizing Fourier integrals, the DeBroglie-Bohr relationships, and the classical Hamiltonian could have been followed. All the machinery for this is, in fact, introduced before the Schroedinger wave equation (time-independent) is written down. This may be a matter of taste, but it would appear to place the postulational basis of wave mechanics in evidence in a clear way.

M. E. ROSE
Oak Ridge National Laboratory

Mechanics and Properties of Matter. REGINALD J. STEPHENSON. Pp. 371+x, Figs. 175. 15×23.5 cm. John Wiley and Sons, Inc., New York, Chapman and Hall, Ltd., London, 1952. Price $6.00.

This book is a new textbook for the intermediate mechanics course. The student is expected to have studied general physics and calculus for at least one year each; the treatment is designed so that the prior knowledge of physics which is assumed is quite modest, and students who have had a thorough two-year course in general physics will find a certain amount of familiar material. The subjects covered are the mechanics of the particle and the rigid body including oscillatory motion and gyroscopic motion and, in addition, there are short treatments of elasticity, hydrostatics and hydrodynamics, and wave motion.

The author has tried to emphasize the physical content of the theory rather than the mathematical formalism. With that purpose the vector algebra has been introduced as needed through the book so that the scalar product appears first in the chapter on work and energy, and the vector product is not introduced until the author is well along in the chapter on rigid body motions. For some of the more complicated subjects the author has given a first approximate treatment by way of introduction before taking up a more thorough-going discussion. Mathematical points with which the student may be unfamiliar are usually explained carefully as in the discussion of conic sections in the chapter on gravitation. There are many illustrative problems whose solutions are given in detail, and each chapter concludes with a large collection of problems of various kinds.

There are several deficiencies in the book which seem important to the reviewer. After a careful discussion of the definition of mass, the author introduces the concept of force without definition and does not discuss the logical status of Newton's second law in his development. The treatment of rotational motion is lengthy, including a discussion of the motion of the top, but the author does not always emphasize the basic physics of the problem. Thus, the vector nature of the angular velocity is not adequately discussed: it is not made clear that although angular velocities are vectors, angular displacements are not. The relationship between angular momentum and angular velocity in not stressed, so that the student may not grasp the basic point that these two vectors are not, in general, parallel to one another. The inertia tensor is mentioned only briefly and on page 216 tensor is misdefined and wrongly exemplified. Finally, there is no discussion of the important subject of coupled oscillations and normal coordinates.

One last point on which the reviewer would like to comment is the author's general approach to the teaching of intermediate mechanics. The author's method is to continue the elementary approach of the general physics course in dealing with the subject matter of this book. The reviewer would prefer a treatment which highlights, to a greater extent, the general ideas and methods of the subject. The student of this book will learn a good deal of mechanics, but he may not acquire an adequate appreciation of the structure, power, and beauty of this science.

MARTIN J. KLEIN
Case Institute of Technology

The Nature of Some of Our Physical Concepts. P. W. BRIDGMAN. Pp. 64. Philosophical Library, New York, 1952. Price $2.75.

The reconstruction of physical concepts which has occurred through theories of relativity and quanta has furthered the establishment of philosophy of science as a special field of knowledge. In recognition thereof a Department of the History and Philosophy of Science was created at City College, University of London, under the leadership of Professor Herbert Dingle; recently there was founded the *British Journal for the Philosophy of Science*. In April, 1950, Professor Bridgman, who has contributed significantly to the analysis of concepts and methods of science, delivered three lectures in London at the invitation of Professor Dingle. These lectures, which were then published in the just cited *British Journal*, have now appeared as the book which is the subject of the present review.

In the *Logic of Modern Physics*, 1927, Professor Bridgman expounded a theory of concepts which under the name of Operationism has risen to the status of a philosophical doctrine. In the present book he endeavors to develop an awareness of the operational content of several of the important concepts of physics. In the first lecture he states that the fundamental idea back of an operational analysis is simple enough, namely, that we do not know the meaning of a concept unless we can specify the operations which were used by us or our neighbor in applying the concept in a concrete situation. The first lecture also discusses the concepts of field and empty space with application to the propagation of light. In the second lecture he examines the concepts of internal energy and entropy in thermodynamics, and especially he finds need for flow vectors and a creation function for entropy. In the third lecture the author examines concepts for the description of electrical and thermal phenomena in massive conductors.

Professor Bridgman's original statement of his operational theory appeared inadequate to specialists in logical analysis. The theory was not formulated in terms of standard distinctions for modes of meaning, such as intension and extension, sense and denotation. Further, inasmuch as rationalist philosophers had stressed the creative element of thought in the constitution of science, Professor Bridgman's apparent emphasis on physical operations seemed to minimize the rational constituent of science. This apparent emphasis on physical operations encouraged psychologists to interpret mental states in terms of modes of behavior. From another point of view, Professor Bridgman's emphatic recourse to immediate experience appeared to imply acceptance of phenomenalism in metaphysics.

In response to discussion of his work, Professor Bridgman subsequently declared that he recognized mental as well as physical operations to be constitutive of meaning. In the present work he stresses mental operations; among

such he singles out for special attention mathematical manipulations which he characterizes as "paper-and-pencil" operations. He further cites verbal operations and emphasizes the function of verbal demands in the development of science. In the present book there is less emphasis on immediate experience, but the language is weighted in favor of empiricism as represented by nominalism in the theory of concepts and behaviorism in psychology.

In his discussion of the concepts of field and empty space, Professor Bridgman argues that there is no instrumental operation by which action at a distance can be distinguished from action through a field. In both cases the propagation of influence is to be detected by a test body. He recognizes, however, that propagation of light by action at a distance would require delay in time. In the reviewer's opinion, such delayed action in empty space would not meet the requirements of the principle of sufficient reason, which principle is satisfied by the determination of a finite velocity by the properties of an intervening medium.

Physical properties essentially are manifested in interactions as dispositional properties which stand in certain relations. Thus, a physical concept signifies a property that is constituted by relations which are expressed by laws. Operations may be required to set the stage for the discovery of a property, but operational procedures merely enable one to isolate a property so that its relation to some other property may be discerned. But this analysis, which was adequately expressed by Helmholtz in *Zaehlen und Messen, erkenntnistheoretisch betrachtet*, 1887, does not contradict operational analysis, which Professor Bridgman characterizes as a necessary but not sufficient condition of meaning. Indeed, he recognizes that the first and second laws of thermodynamics furnish definition of internal energy and entropy. He finds that by instrumental supplementation a definition develops into a natural law; the reviewer, however, would argue that initially the transition occurs in the reverse direction. Regardless of this question, acknowledgement that functional relations may serve as definitions of concepts implies that functional relation constitutes meaning.

V. F. LENZEN
University of California, Berkeley

Symmetry. HERMANN WEYL. Pp. 167, Figs. 72, 14.5×22.5 cm. Princeton University Press, Princeton, New Jersey, 1952. Price $3.75.

Symmetry is not only basic in the forms which man creates but it is one of the ideas by which he tries to understand the order and relations of nature. This little book aims to show the wide application of this idea as it cuts across various fields and it further aims, after making the definition precise, to outline its mathematical essence.

Any writing by so distinguished a mathematician as Professor Weyl must deserve attention, and here is a semi-popular summary of the work of a penetrating mathematical mind in viewing the world through the one small but fascinating window of symmetry. The volume has been described by Weyl himself as a "swan song on the eve of my retirement from the Institute of Advanced Study." Is there an intrinsic difference between right and left? Can a uniform hexagonal network be made to cover a sphere? Are all *a priori* statements in physics based on symmetry? Weyl answers these questions and many others.

The book is richly illustrated with many striking examples from the arts and from organic and inorganic nature. These range from the cells of a beehive to living cell division, from the geometric intricacies of Arabian ornament to Laue spots, from the mosaics of Ravenna and from Sumerian and Babylonian design to the triple pole symmetry of the iris and the hexagonal symmetry of the snow flake.

The four chapters are a slight revision of the Louis Clark Vanuxem Lectures given at Princeton University in 1951. The first two lectures discuss the subjects of (1) *Bilateral Symmetry*, and (2) *Translational, Rotational and Related Symmetries* with illustrations chiefly from the arts and biology. The third lecture on *Ornamental Symmetry* begins a somewhat more systematic attempt to analyze the particular kind of geometric symmetry which in two dimensions leads to the seventeen possible types of symmetry involved in ornament. In the last lecture this line of thought is extended to the three-dimensional case basic to crystallographic analysis. The expressed aim in this lecture is to exhibit the principle of symmetry in both physics and mathematics and to arrive at a final general statement (mathematical) of that principle.

The theories of relativity and quanta are referred to in the final lecture, the former particularly for the symmetry of the transformations of the Lorentz group, and the latter through brief reference to the kinds of symmetry involved in atomic structure and spectra.

There is much for the nonmathematical reader who may also glimpse the workings of the mathematical mind. Detailed steps are omitted, but the final mathematical conclusions would be clearer to those with some knowledge of modern geometry. Many readers will benefit more from seeing how the principle applies in fields other than their own. There are two appendices containing mathematical proofs.

ROGERS D. RUSK
Mount Holyoke College

New Members of the Association

The following persons have been made members or junior members (*J*) of the American Association of Physics Teachers since the publication of the preceding list [Am. J. Phys. 20, 598 (1952)].

Alexander, William Everett (*J*), R.R. 6, Marion, Ind.
Bade, William Lemoine, Brace Laboratory, University of Nebraska, Lincoln, Neb.
Basye, Clifton A., 103 Bellevue Dr., Richmond, Ky.
Bounds, Nell M., Midwestern University, Wichita Falls, Texas
Bradley, George Edgar, Western Michigan College, Kalamazoo, Mich.
Dittrich, William J., 515 Lakeway Dr., Bellingham, Wash.
Dunkerley, Herbert Spencer (*J*), 102 N. Coler St., Urbana, Ill.

Hart, Edward Hutson, 1055 Pine St., Menlo Park, Calif.
Hawley, Fred T., 187 Hamilton, Muskegon, Mich.
Jewell, Paul Vernon, Maryland State College, Princess Anne, Md.
Johnson, Carl Valdemar, 285 Main St., Portland, Conn.
Johnson, Halbert Herring, 7346 Arnim, Houston 17, Texas
Kriessman, Charles Joseph, 1411 Viers Mill Rd., Rockville, Md.
Leonard, Robert S. (*J*), P.O. Box 9342, University Station, Reno, Nev.
Marion, Jerry Bascervil (*J*), Physics Department, Rice Institute, Houston 1, Texas
Morley, Arthur L., Rt. 5, Hendersonville, N.C.
Pollack, Edward (*J*), 681 E. 181 St., New York 57, N. Y.
Rapp, Thomas Louis (*J*), Apt. B-8-B College View, College Station, Texas
Schrader, Carlton Davis, Physics Department, State University of Iowa, Iowa City, Iowa
Schriever, William W., Jr., 910 Chautauqua, Norman, Okla.
Stecher, Milton, 916 Union St., Brooklyn 15, N. Y.
Stephenson, Harold P., Illinois Wesleyan University, Bloomington, Ill.
Stubbs, George R., 3126 23rd Ave., S., Minneapolis 7, Minn.
Vernon, Robert Carey, Department of Physics, Williams College, Williamstown, Mass.
White, James Frederick, 1949 Woodard St., Abilene, Texas.

AMERICAN JOURNAL OF PHYSICS VOLUME 21, NUMBER 1 JANUARY, 1953

RECENT MEETINGS

Western Pennsylvania Section

The Western Pennsylvania Section of the American Association of Physics Teachers met at Carnegie Institute of Technology on December 1, 1951. Forty-three members and guests were present. PROFESSOR CHARLES WILLIAMSON, *Carnegie Institute of Technology*, presided. Dr. J. C. WARNER, president of Carnegie Tech, welcomed the members and presented an invited paper entitled **The present status of the scientific manpower problem.** The following contributed papers were presented:

Discussion of the report of the Coulomb's Law Committee of the AAPT. W. H. MICHENER, *Allegheny College*. The history of the committee and the background of the report were given. Some of the important recommendations of the committee were explained. The speaker believes that, in the teaching of magnetism, the traditional model of magnet poles will gradually be supplanted by something like the Amperian current theory. A coming generation of physicists may be as unfamiliar with Coulomb's law for magnets as our present younger generation is with stresses in the ether.

A decade scaler using neon tubes. MERLE N. HIRSH, *University of Pittsburgh*. A decade scaler using neon tubes is now in development as an undergraduate thesis project. Based on a circuit by Manley and Buckley,[1] it offers economy of power consumption and panel space, dependability at high counting speeds, and relative insensitivity to variation in supply voltage. To increase dependability, but not initial cost, we have replaced the germanium diodes of the original circuit with 6AL5 double diodes, one tube to two counting stages. Eventual counting speeds of 1500 cps should be possible.

[1] J. Manley and E. Buckley, Electronics **23**, 84 (1950).

Teaching the connection between the physical world and abstract mathematical analysis. S. DE BENEDETTI, *Carnegie Institute of Technology*. The mind of the student should be kept focused on what happens and not on what is written in textbooks. It is important that, chiefly at the beginning (mechanics), problems should be of an experimental nature and involve some measurement. As a second step the student should be trained in the process of abstraction which leads from the direct experience with a concrete physical situation to the formulation of a "model" capable of mathematical analysis. Only when the student is thoroughly familiar with this process should one introduce "metaphysical" entities such as booms 5 ft long (exactly), weights of 100 lb (exactly), frictionless planes, and weightless strings.

Wanted: Generic term for a class of units. CECIL O. RIGGS, *Waynesburg College*. There seems to be no name to distinguish the correct unit for any quantity from auxiliary units of the same quantity. For example, where an equation requires or gives a result in erg rather than joule, ft-lb per sec rather than horsepower, etc., there is no general statement with which the student can be warned. The author of one textbook uses the term fundamental unit for the proper unit and derived unit for the auxiliary unit, but these terms have of course been pre-empted for use in another sense. Some writers seem to use the term, absolute unit, for the proper unit, but this is ambiguous as absolute unit is usually used for any unit of an absolute system. Common agreement upon a general term would be a convenience.

Science is fun at the Buhl Planetarium. RICHARD C. HITCHCOCK, *Buhl Planetarium*. Our primary purpose is to show science principles to people of all ages. For young audiences, school work is supplemented rather than sup-

planted. For adults, the science of everyday living is stressed. It is essential that exhibits be safe. In addition, action exhibits must have sturdy mechanical construction to permit continued operation. This latter is most important, as a museum may be regarded as a laboratory run by inexperienced students. All-day, every-day operation is a real problem for the exhibit designer and the maintenance staff.

The Muller microscope. BERNARD L. BRINKER, *St. Vincent College, Latrobe, Pennsylvania.* With blackboard sketches and verbal descriptions, the Muller microscope was reported to the meeting. This new instrument is capable of very high resolution (about 0.3 millimicrons) which allows the imaging of details in the crystal structure of metals down to about atomic diameters. The microscope was devised by Dr. Erwin Muller of the Kaiser Wilhelm Institute at Berlin, Germany. It was demonstrated by Dr. Muller at the Ninth Annual Pittsburgh Conference on X-ray and Electron Diffraction (November 30, 1951). With the instrument and a projection lens, the structural patterns of tungsten and barium crystals and the crystallizing process itself were shown at the conference.

The "cycle" in a physical interpretation of Planck's constant (h) for circular dynamics. BERNARD L. BRINKER, *St. Vincent College, Latrobe, Pennsylvania.* In an analysis of periodic movement, the cycle or its equivalent has physical significance. It cannot be omitted from an expression without changing the physical concept explicitly stated. By an explicit transformation of the physical quantities involved in Planck's constant ($h=6.62\times10^{-27}$ erg sec/cycle), an equivalent torque-impulse ($h=1.05\times10^{-27}$ cm dyne sec) can readily be derived for circular (orbital) dynamics. Such a conceptual transformation had been implied by Dirac's introduction of $\hbar=h/2\pi$ for quantum mechanics generally.

At the business meeting in the afternoon, the group voted to change the name of the organization to "Western Pennsylvania Section of the American Association of Physics Teachers." New officers are President, W. C. KELLY, *University of Pittsburgh;* Vice-president, R. L. BROWN, *Allegheny College;* Secretary, R. C. HITCHCOCK, *Buhl Planetarium.* The secretary read the reply of the Pennsylvania State Council of Education to the recommendation of the Section that certification requirements for high school physics teachers include at least 6 semester hours of college physics. The Council feels that the present supply situation makes it unwise to take action that would result in more difficulty for school districts to obtain teachers and therefore withheld approval of the recommendation. The Section voted to continue an interest in this matter by including an item in the minutes of future meetings as a reminder for future action.

FATHER BERNARD L. BRINKER, the national representative of the Section, reported on proceedings of the business section of AAPT at its last New York meeting. He also called attention to the annual meetings of the Pennsylvania Physics Conference. The Section instructed the secretary to send a letter of sympathy to the family of DR. W. N. ST. PETER. A vote of thanks was given to the Carnegie Tech members for their hospitality. The spring meeting will be held at Allegheny College on May 10, 1952, and the fall meeting at Buhl Planetarium in December.

W. C. KELLY,
Secretary

How to ſecure Houſes, &c. from LIGHTNING

It has pleaſed God in his Goodneſs to Mankind, at length to diſcover to them the Means of ſecuring their Habitations and other Buildings from Miſchief by Thunder and Lightning. The Method is this: Provide a ſmall Iron Rod (it may be made of the Rod-iron uſed by the Nailers) but of ſuch a Length, that one End being three or four Feet in the moiſt Ground, the other may be ſix or eight Feet above the higheſt Part of the Building. To the upper End of the Rod faſten about a Foot of Braſs Wire, the Size of a common Knitting-needle, ſharpened to a fine Point; the Rod may be ſecured to the Houſe by a few ſmall Staples. If the Houſe or Barn be long, there may be a Rod and Point at each End, and a middling Wire along the Ridge from one to the other. A Houſe thus furniſhed will not be damaged by Lightning, it being attracted by the Points, and Paſſing thro the Metal into the Ground without hurting any Thing. Veſſels alſo, having a ſharp pointed Rod fix'd on the Top of their Maſts, with a Wire from the Foot of the Rod reaching down, round one of the Shrouds, to the Water, will not be hurt by Lightning.—Poor Richard's Almanac, 1753.

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